Engineering Fluid Mechanics

(NCE 201) LTPC 3-1-0-4

2nd B.Tech. Civil Engineering

Dr. Deepesh Singh
Department of Civil Engineering

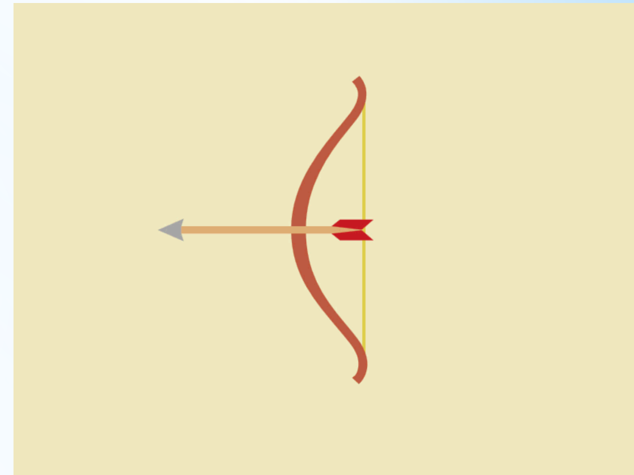
Harcourt Butler Technical University, Kanpur

*Introduction

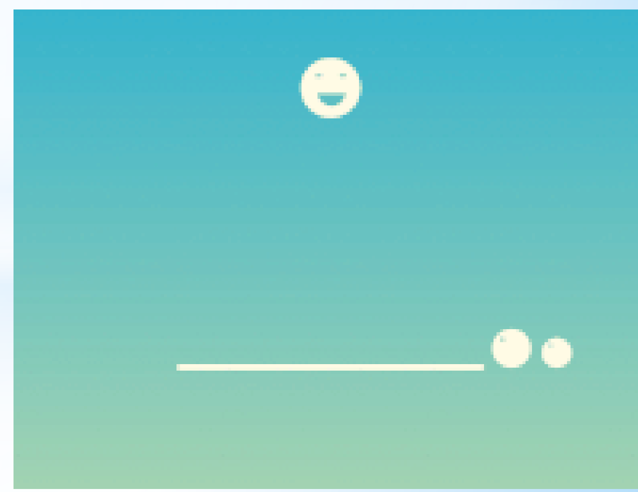
Fluid mechanics deals with liquids and gases in motion or at rest.

*Introduction

- The branch of mechanics that deals with bodies at rest is called **statics**,

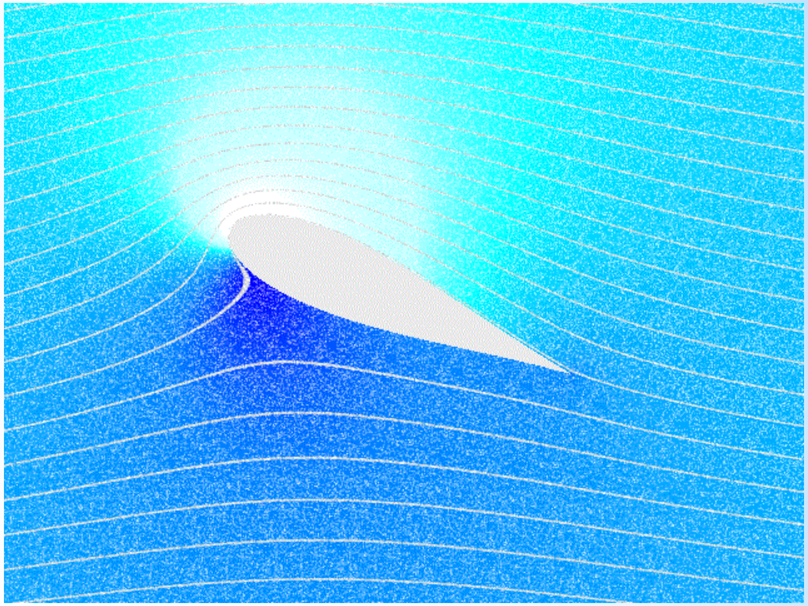


- While the branch that deals with bodies in motion is called **dynamics**.



*Introduction

* The subcategory **fluid mechanics** is defined as the science that deals with the behavior of fluids at rest (*fluid statics*) or in motion (*fluid dynamics*), and the interaction of fluids with solids or other fluids at the boundaries.



*Introduction

* Fluid mechanics is also referred to as **fluid dynamics** by considering fluids at rest as a special case of motion with zero velocity (Fig. 1–1).

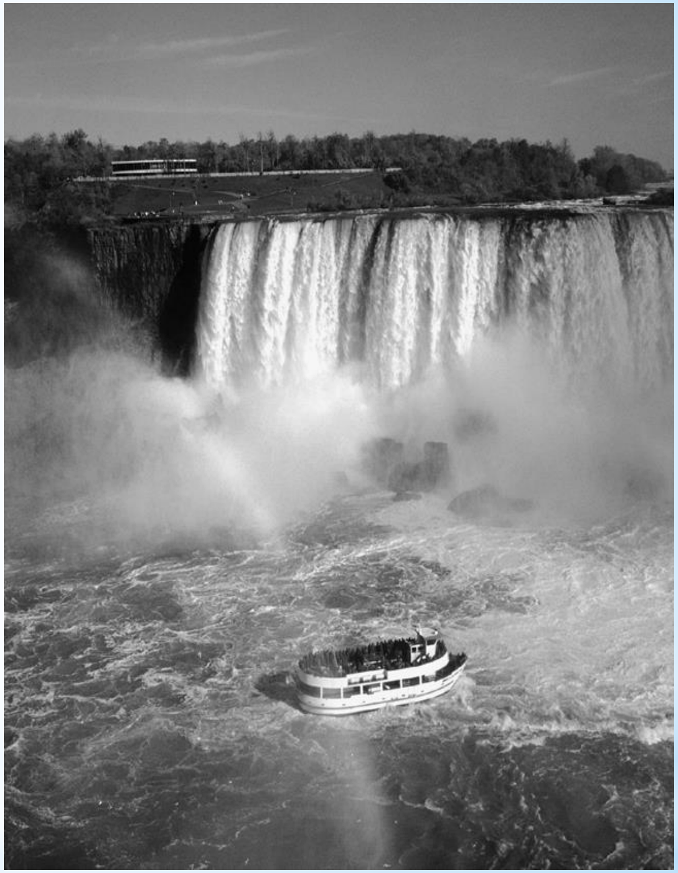


FIGURE 1-1 Fluid mechanics deals with liquids and gases in motion or at rest.

*Introduction

Fluid mechanics itself is also divided into several categories. The study of

* **Hydrodynamics**: the motion of fluids that are practically incompressible (such as liquids, especially water, and gases at low speeds) is usually referred to as.

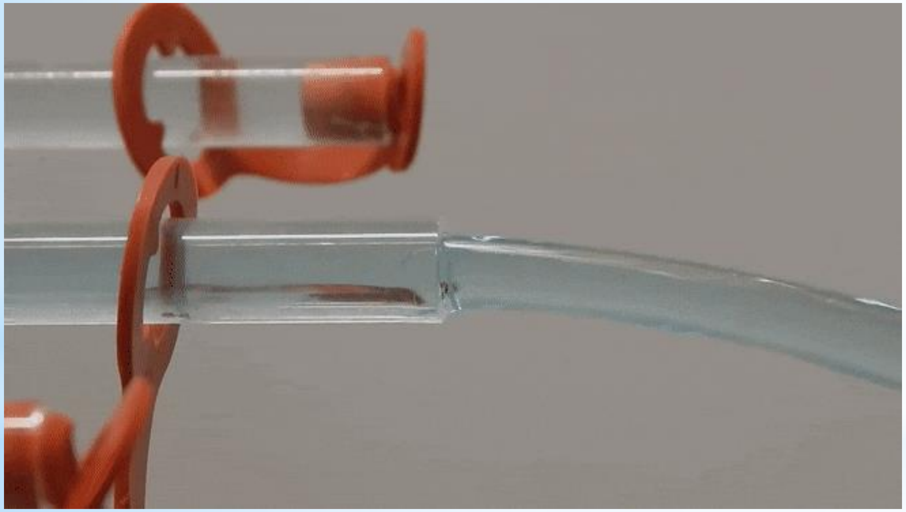
Incompressible fluid

Compressible fluid

*Introduction

Fluid mechanics itself is also divided into several categories. The study of

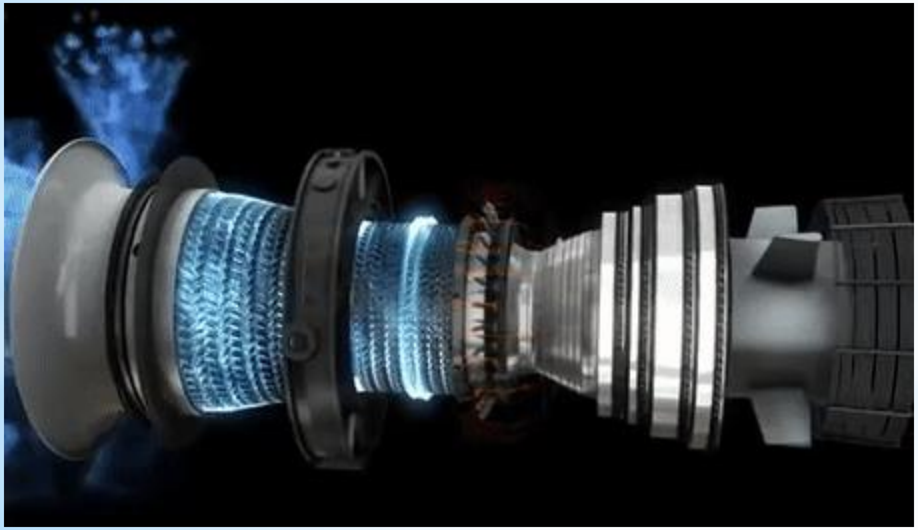
* A subcategory of hydrodynamics is **hydraulics**, which deals with liquid flows in pipes and open channels.

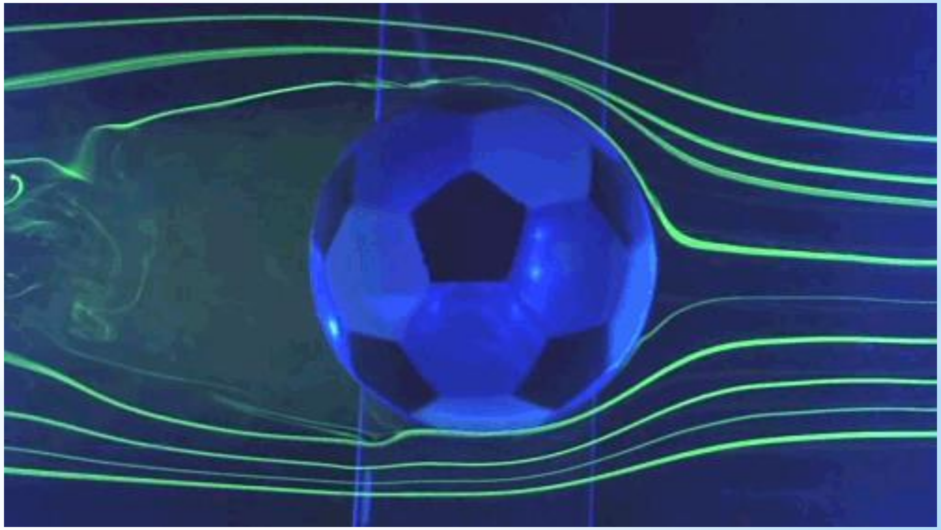


*Introduction

Fluid mechanics itself is also divided into several categories. The study of

* **Gas dynamics** deals with the flow of fluids that undergo significant density changes, such as the flow of gases through nozzles at high speeds.

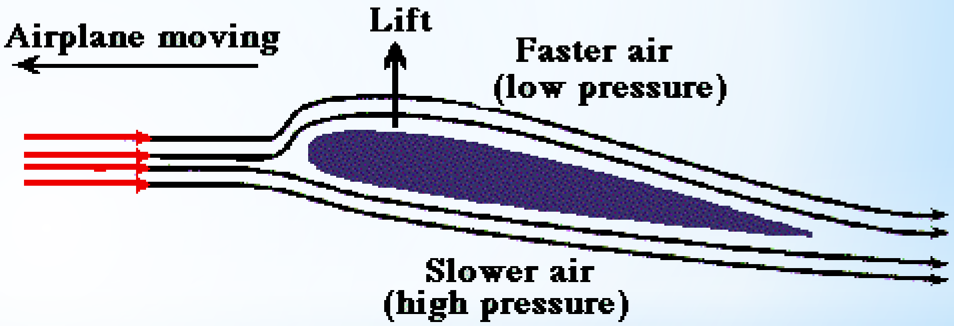




*Introduction

Fluid mechanics itself is also divided into several categories. The study of

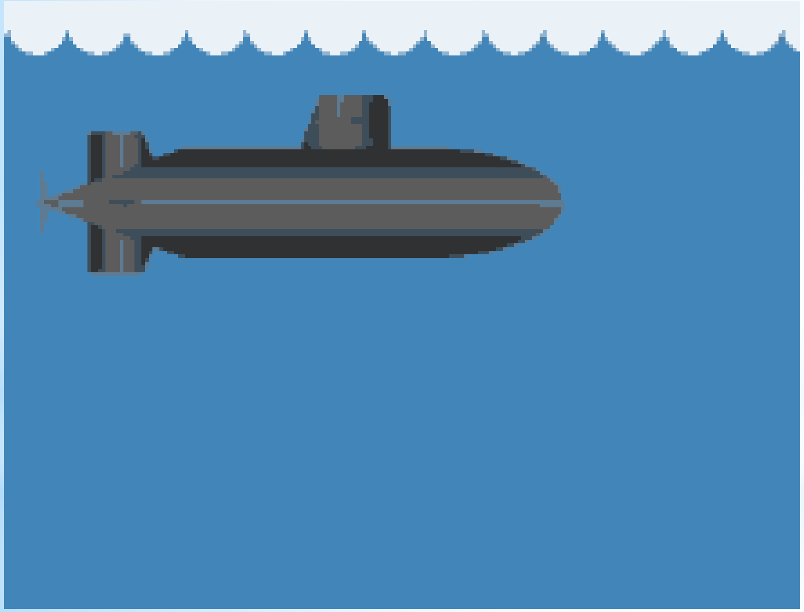
* **Aerodynamics** deals with the flow of gases (especially air) over bodies such as aircraft, rockets, and automobiles at high or low speeds.



*Introduction

Fluid mechanics itself is also divided into several categories. The study of

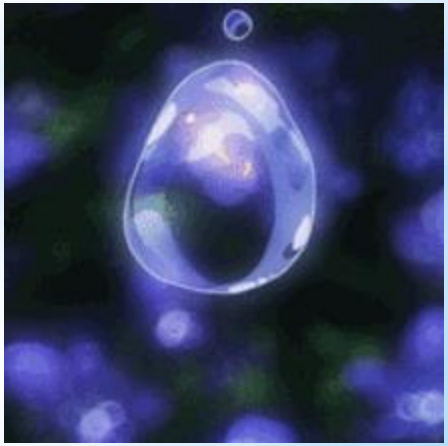
* Some other specialized categories such as **meteorology, oceanography,** and **hydrology** deal with naturally occurring flows.



What is a fluid?

* FLUID: is a substance which is capable of flowing

* FLUID: is a substance which deforms continuously when subjected to external shearing force.



*What is a fluid?

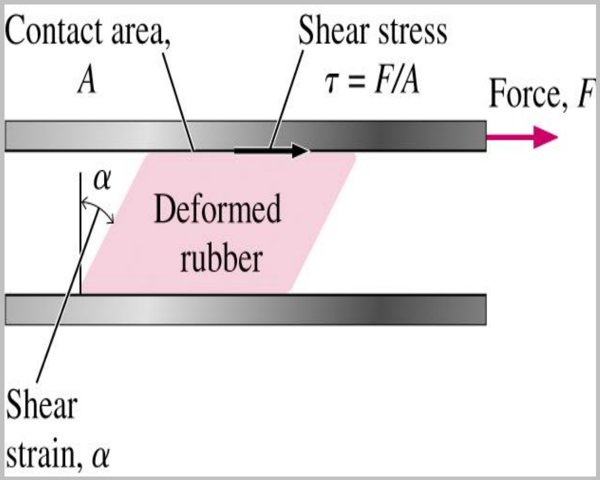
* A substance in the liquid or gas phase is referred to as a **fluid.**
* Distinction between a solid and a fluid is made on the basis of the substance’s ability to resist an applied shear (or tangential) stress that tends to change its shape.
* A solid can resist an applied shear stress by deforming, whereas a fluid deforms continuously under the influence of shear stress, no matter how small.
* In solids stress is proportional to *strain*, but in fluids stress is proportional to *strain rate*. When a constant shear force is applied, a solid eventually stops deforming, at some fixed strain angle, whereas a fluid never stops deforming and approaches a certain rate of strain.

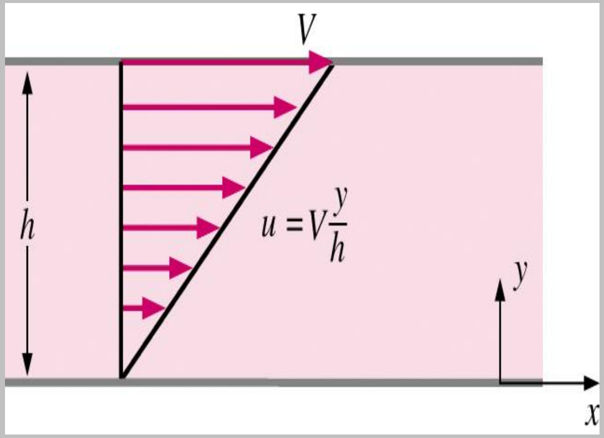
*What is a fluid?

- *Distinction between solid and fluid?
  - *Solid: can resist an applied shear by deforming. Stress is proportional to strain
  - *Fluid: deforms continuously under applied shear. Stress is proportional to strain rate

Solid

$$\tau = \frac{F}{A} \propto \alpha$$

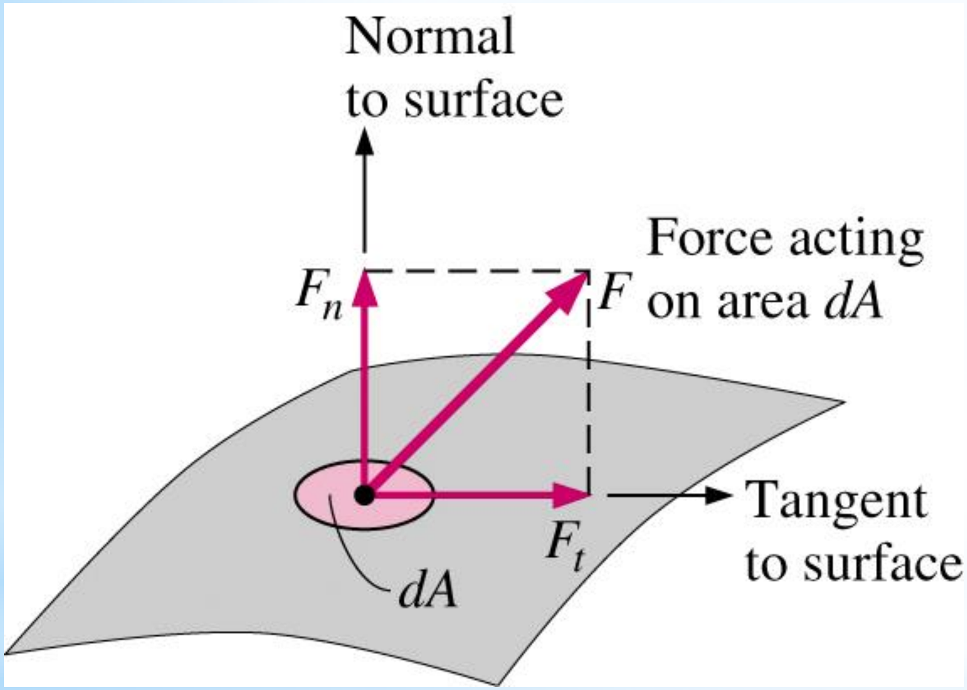




Fluid

$$\tau = \frac{F}{A} \propto \mu \frac{V}{h}$$

*What is a fluid?



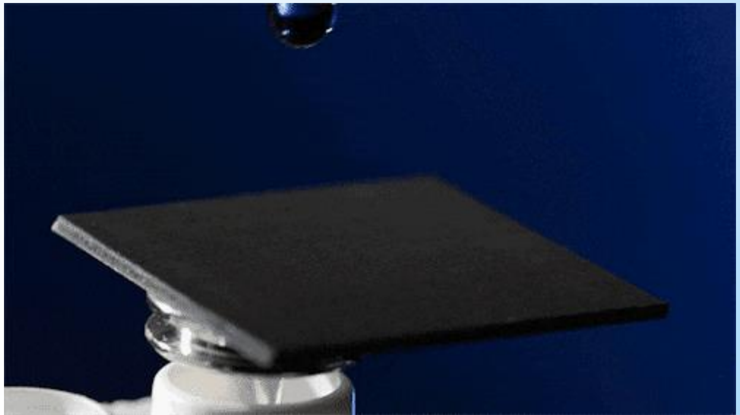
- *Stress is defined as the force per unit area.
- *Normal component: normal stress
  - *In a fluid at rest, the normal stress is called **pressure**
- *Tangential component: shear stress

*What is a fluid?

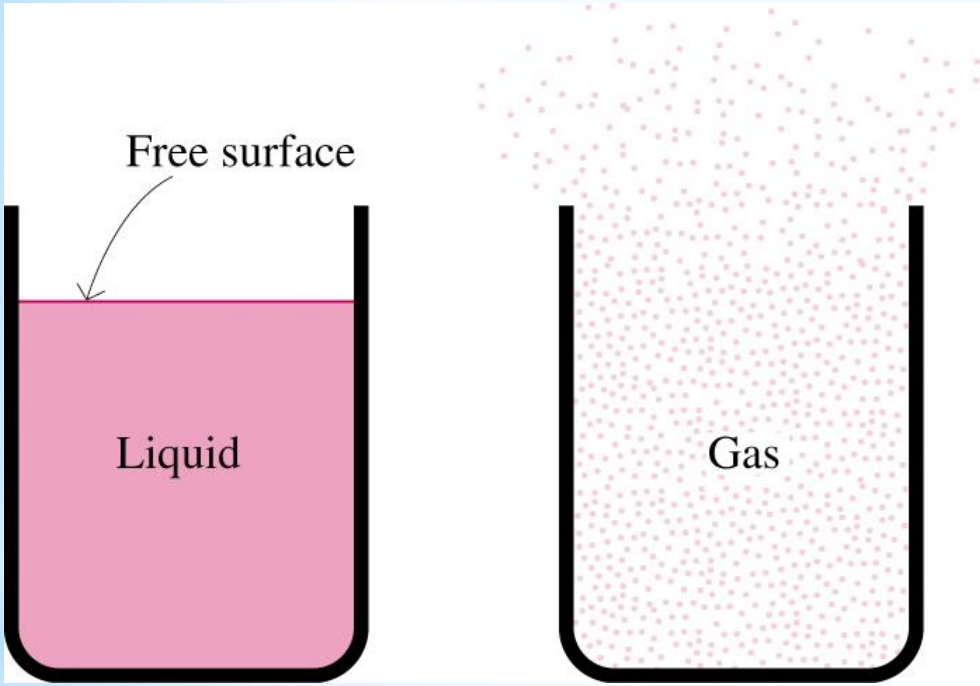
*FLUID HAS FOLLOWING CHARACTERISTICS:

1. NO DEFINITE SHAPE: IT CONFORMS THE SHAPE OF A CONTAINING VESSEL

2. EVEN A SMALL AMOUNT OF SHEARING FORCE: CAN CAUSE DEFORMATION WHICH CONTINUES AS LONG AS FORCE CONTINUES TO BE APPLIED



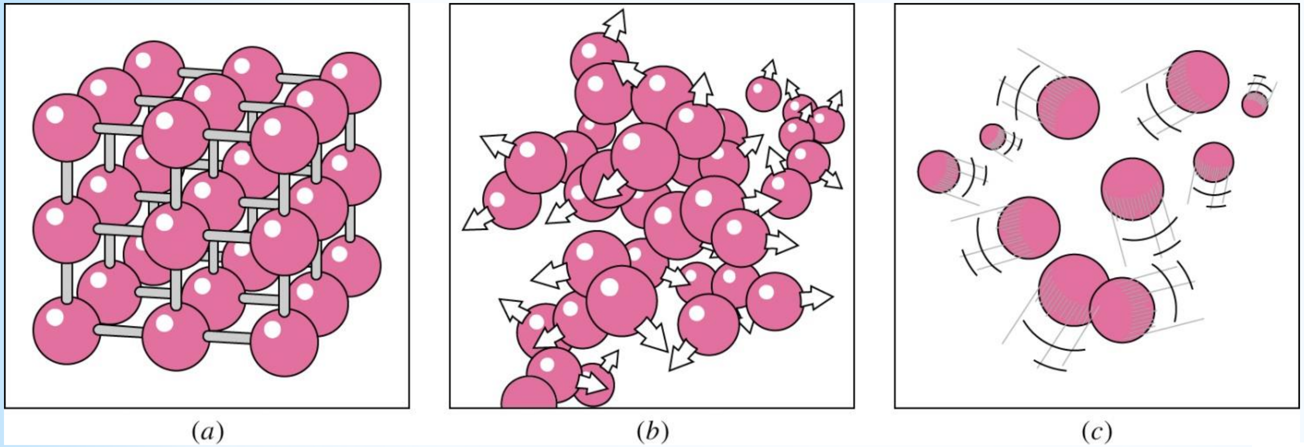
*What is a fluid?



* A liquid takes the shape of the container it is in and forms a free surface in the presence of gravity
* A gas expands until it encounters the walls of the container and fills the entire available space. Gases cannot form a free surface
* Gas and vapor are often used as synonymous words

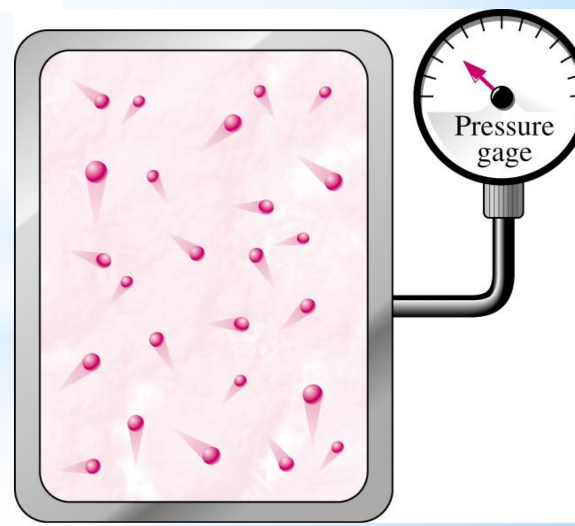
*What is a fluid?

A



solid liquid Vapour/ gas

On a microscopic scale, pressure is determined by the interaction of individual gas molecules.



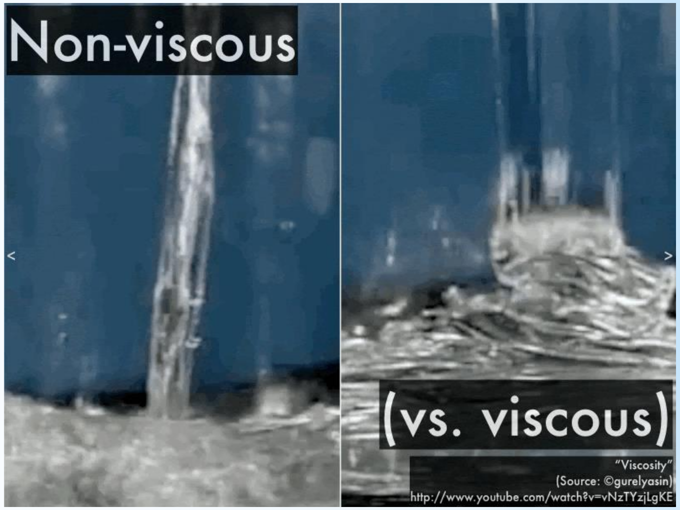
Intermolecular bonds are strongest in solids and weakest in gases. One reason is that molecules in solids are closely packed together, whereas in gases they are separated by relatively large distances

Vapour: It is a GAS whose temp. and press are such that it is very near to liquid state e.g. STEAM

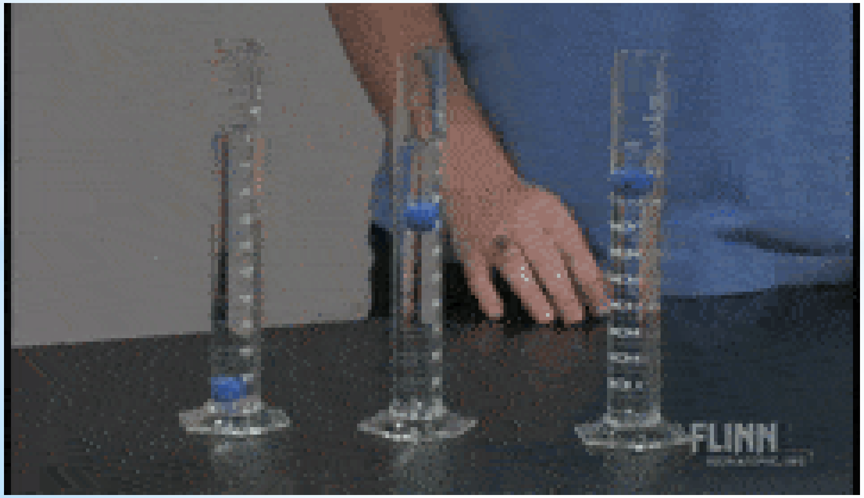
B

*What is a fluid?

1. IDEAL Fluid: NO VISCOSITY, NO SURFACE TENSION, INCOMRESSIBLE eg. WATER AND AIR- USED TO SIMPLIFY MATHEMATICS- NOT IN NATURE

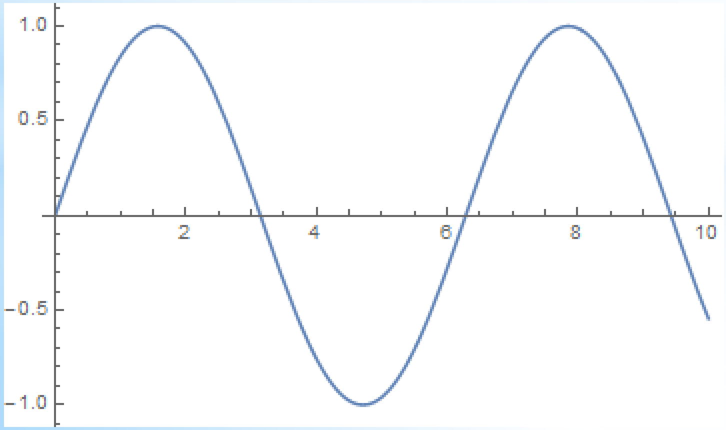


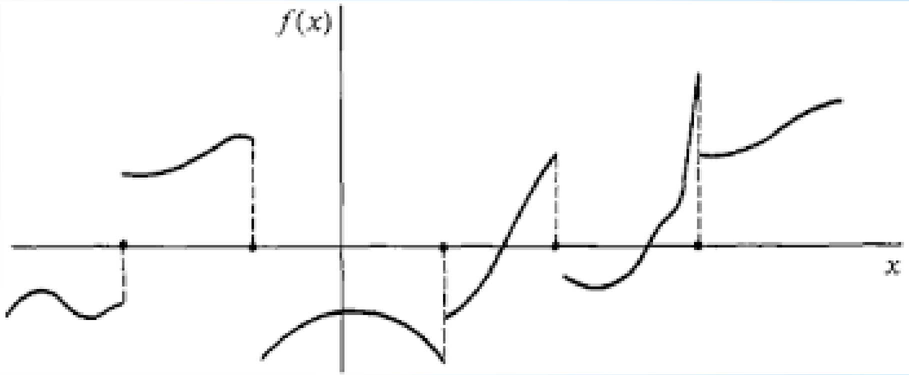
2. REAL Fluid: ALL FLUIID PRESENT IN NATURE



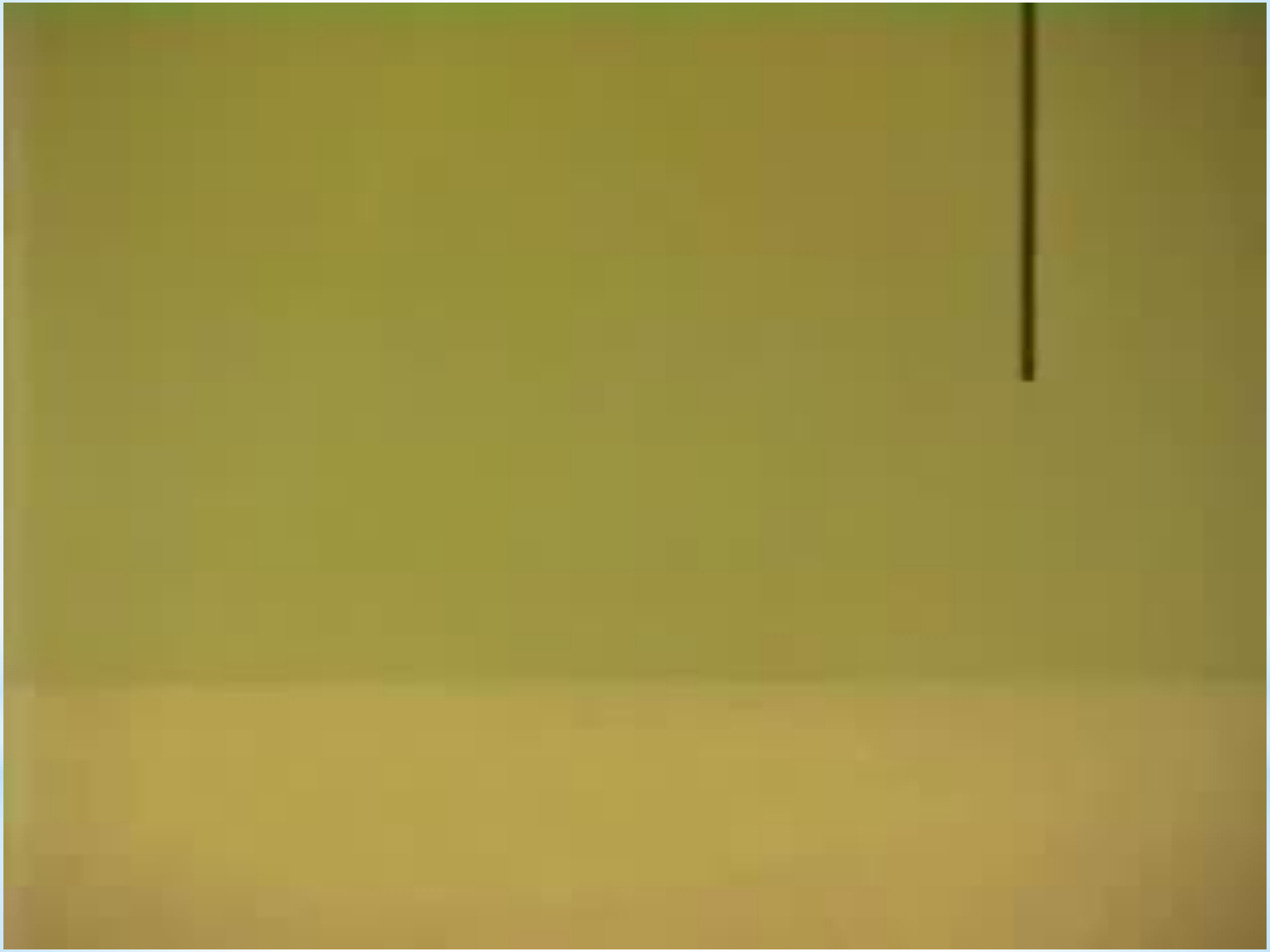
* Fluid is Continuum

A continuous and homogeneous medium/ continuous distribution of mass within the matter or system with no empty space/ fluid properties such as density, viscosity, thermal conductivity, temperature etc. can be expressed as continuous function of space and time

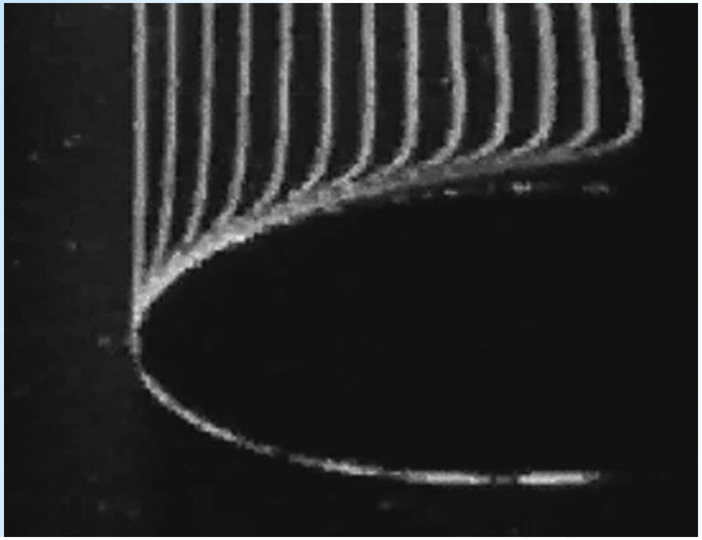


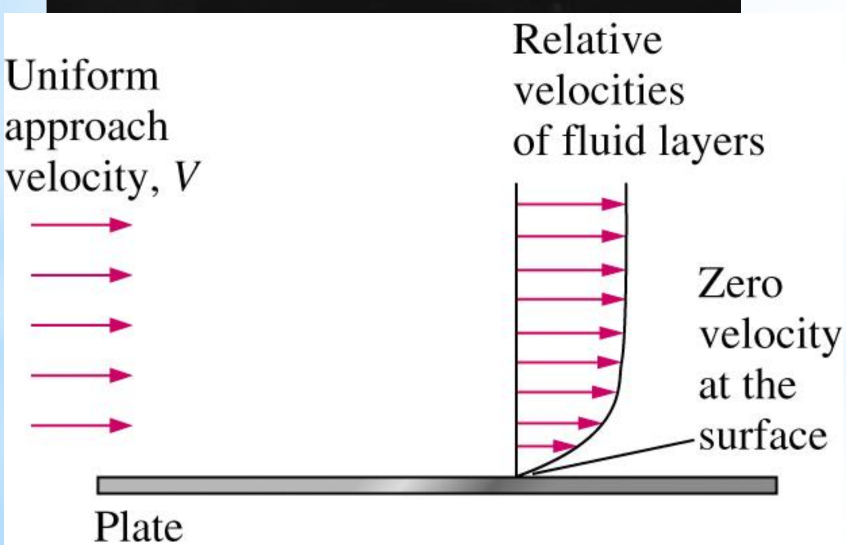


*No-slip condition



No-slip condition

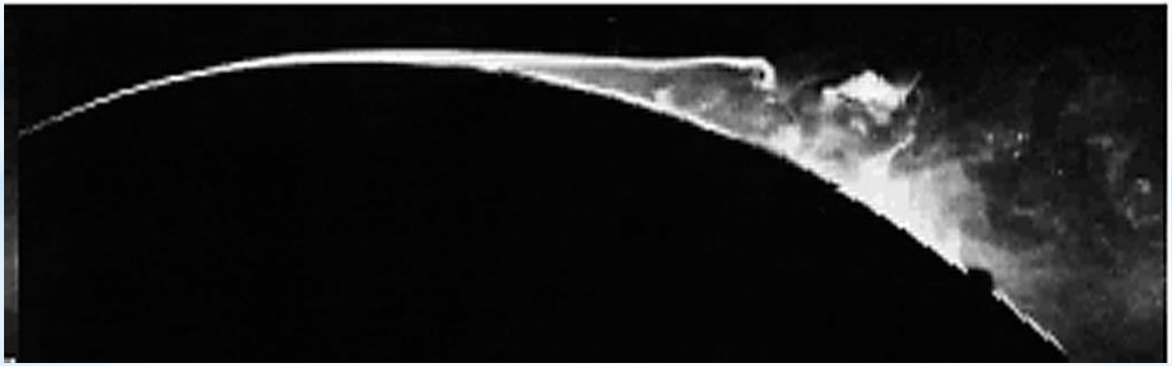




- No-slip condition: A fluid in direct contact with a solid ``sticks'' to the surface due to viscous effects
- Responsible for generation of wall shear stress τ_w, surface drag D= $\int \tau_w \, dA$, and the development of the boundary layer
- The fluid property responsible for the no-slip condition is **viscosity**
- Important boundary condition in formulating initial boundary value problem (IBVP) for analytical and computational fluid dynamics analysis

*No-slip condition

When a fluid is forced to flow over a curved surface, the boundary layer can no longer remain attached to the surface, and at some point it separates from the surface—a process called **flow separation**. We emphasize that the no-slip condition applies *everywhere* along the surface, even downstream of the separation point. Flow separation is discussed in greater detail in Chap. 10.



* A BRIEF HISTORY OF FLUID MECHANICS

Please refer to section 1-3 in the text book

From 283 to 133 BC, they built a series of pressurized lead and clay pipelines, up to 45 km long that operated at pressures exceeding 1.7 MPa (180 m of head)
Done at the Hellenistic city of Pergamon in present-day Turkey.

*Classification of Flows

* We classify flows as a tool in making simplifying assumptions to the governing partial-differential equations, which are known as the Navier-Stokes equations
  * Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0$$

  * Conservation of Momentum

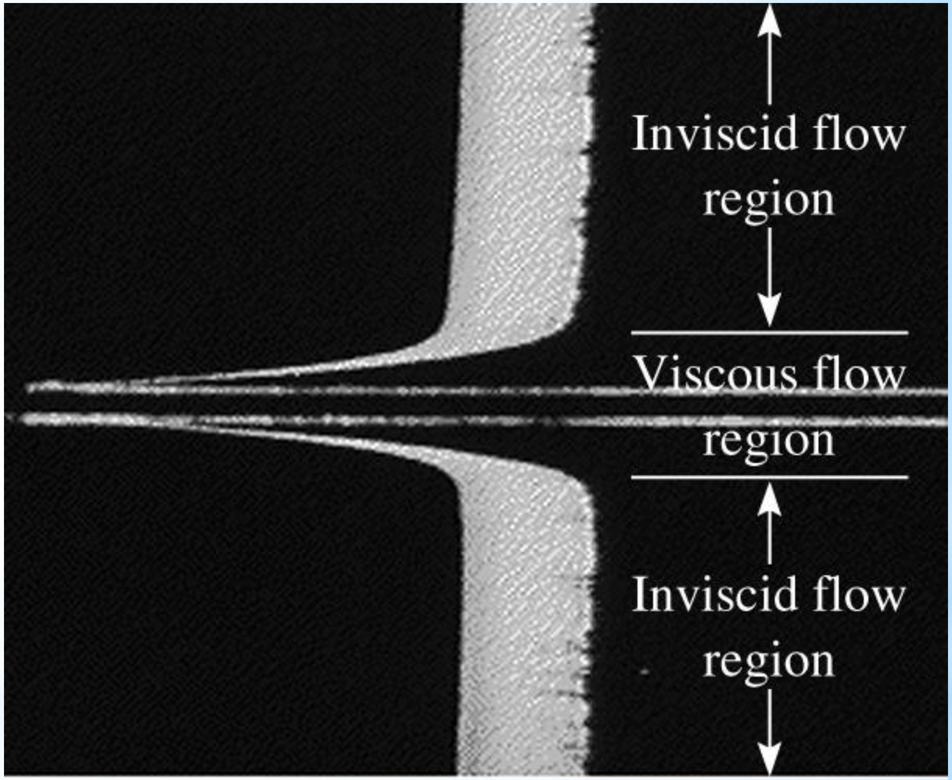
$$\rho \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{U}$$

Viscous vs. Inviscid Regions of Flow

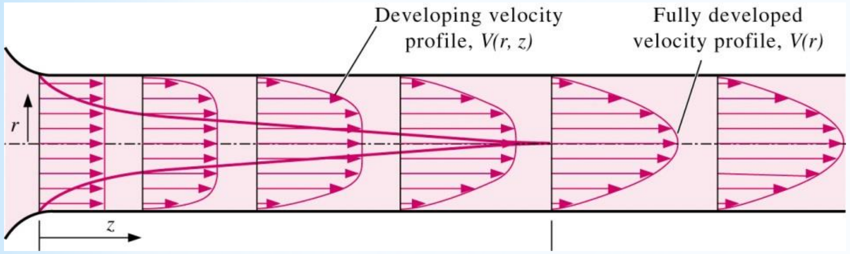
- Regions where frictional effects are significant are called viscous regions. They are usually close to solid surfaces.
- Regions where frictional forces are small compared to inertial or pressure forces are called inviscid

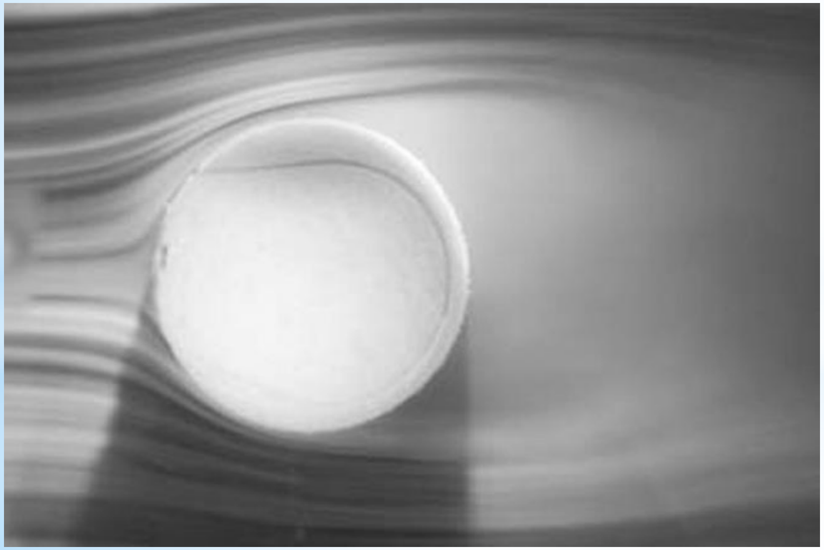
For inviscid flows:

$$\rho \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\,\mathbf{U} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{U} \quad (\mu \nabla^2 \mathbf{U} \to 0)$$



*Internal vs. External Flow





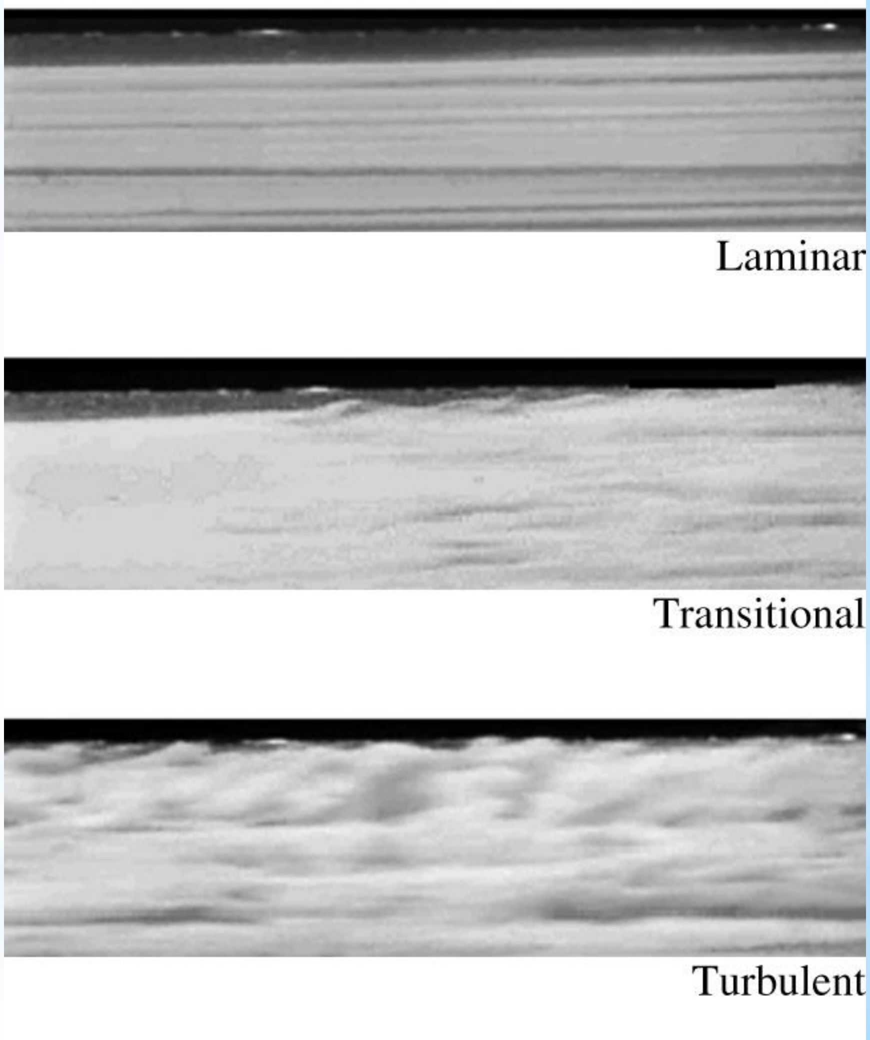
- *Internal flows are dominated by the influence of viscosity throughout the flowfield
- *For external flows, viscous effects are limited to the boundary layer and wake.

Compressible vs. Incompressible Flow

- A flow is classified as incompressible if the density remains nearly constant.
- Liquid flows are typically incompressible.
- Gas flows are often compressible, especially for high speeds.
- Mach number, $Ma = V/c$ is a good indicator of whether or not compressibility effects are important.
  - $Ma < 0.3$: Incompressible
  - $Ma < 1$: Subsonic
  - $Ma = 1$: Sonic
  - $Ma > 1$: Supersonic
  - $Ma >> 1$: Hypersonic

*Laminar vs. Turbulent Flow

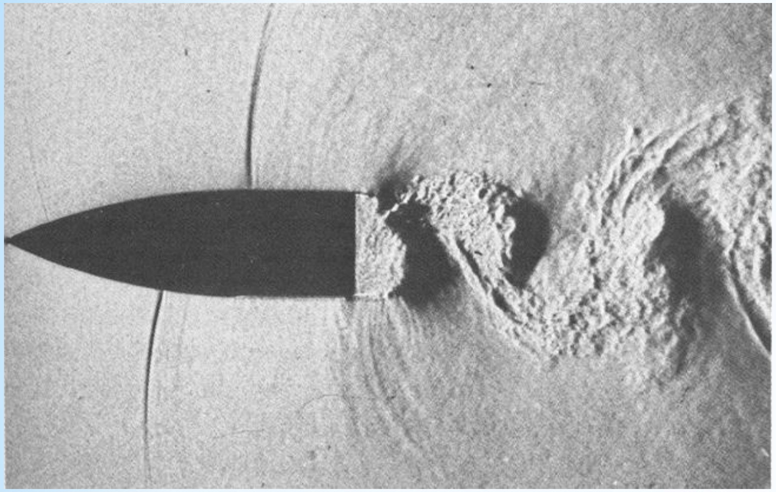
* Laminar: highly ordered fluid motion with smooth streamlines.
* Turbulent: highly disordered fluid motion characterized by velocity fluctuations and eddies.
* Transitional: a flow that contains both laminar and turbulent regions
* Reynolds number, Re= $\rho UL/\mu$ is the key parameter in determining whether or not a flow is laminar or turbulent.

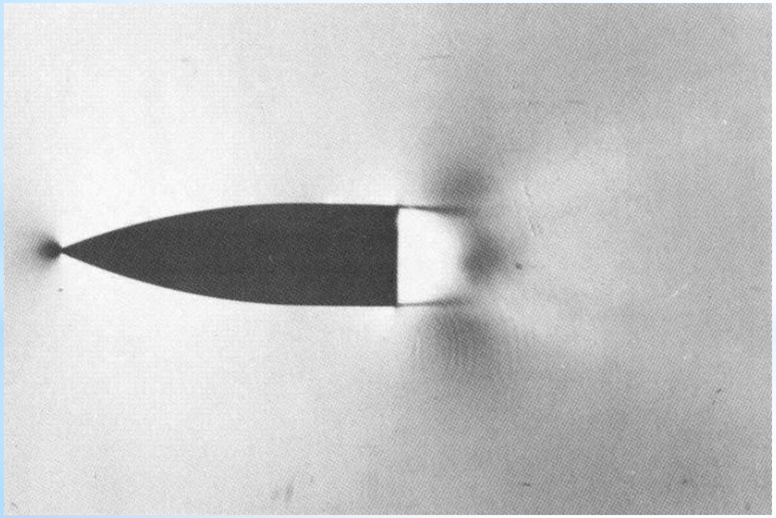


* Natural (or Unforced) versus Forced Flow

* A fluid flow is said to be natural or forced, depending on how the fluid motion is initiated.
* In **forced flow,** a fluid is forced to flow over a surface or in a pipe by external means such as a pump or a fan.
* In **natural flows,** any fluid motion is due to natural means such as the buoyancy effect, which manifests itself as the rise of the warmer (and thus lighter) fluid and the fall of cooler (and thus denser) fluid

*Steady vs. Unsteady Flow

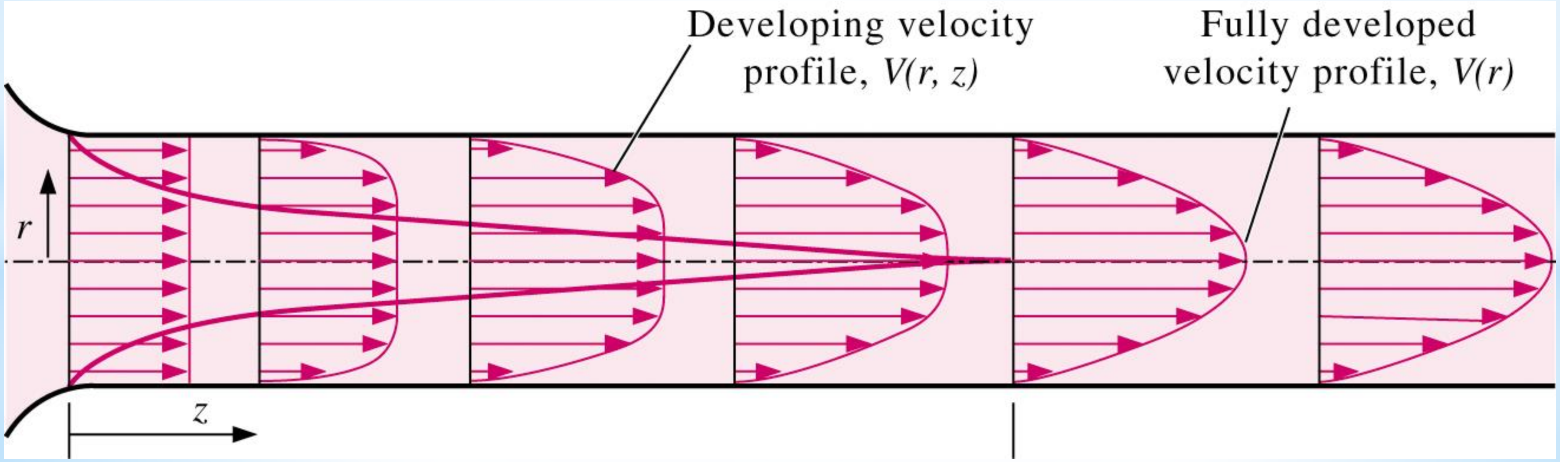




* Steady implies no change at a point with time. Transient terms in N-S equations are zero $\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial \rho}{\partial t} = 0$
* Unsteady is the opposite of steady.
  * Transient usually describes a starting, or developing flow.
  * Periodic refers to a flow which oscillates about a mean.
* Unsteady flows may appear steady if "time-averaged"

* One-, Two-, and Three-Dimensional Flows

* N-S equations are 3D vector equations.
* Velocity vector, $\mathbf{U}(x,y,z,t) = [U_x(x,y,z,t), U_y(x,y,z,t), U_z(x,y,z,t)]$
* Lower dimensional flows reduce complexity of analytical and computational solution
* Change in coordinate system (cylindrical, spherical, etc.) may facilitate reduction in order.
* Example: for fully-developed pipe flow, velocity *V(r)* is a function of radius *r* and pressure *p(z)* is a function of distance *z* along the pipe.



* One-, Two-, and Three-Dimensional Flows

A flow may be approximated as *two-dimensional* when the aspect ratio is large and the flow does not change appreciably along the longer dimension.
For example, the flow of air over a car antenna can be considered two-dimensional except near its ends since the antenna's length is much greater than its diameter, and the airflow hitting the antenna is fairly uniform

One-, Two-, and Three-Dimensional Flows

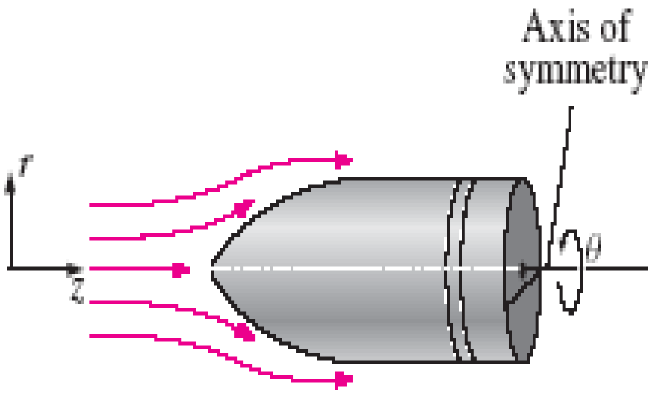


FIGURE 1–22
Axisymmetric flow over a bullet.

EXAMPLE 1–1 Axisymmetric Flow over a Bullet

Consider a bullet piercing through calm air. Determine if the time-averaged airflow over the bullet during its flight is one-, two-, or three-dimensional (Fig. 1–22).

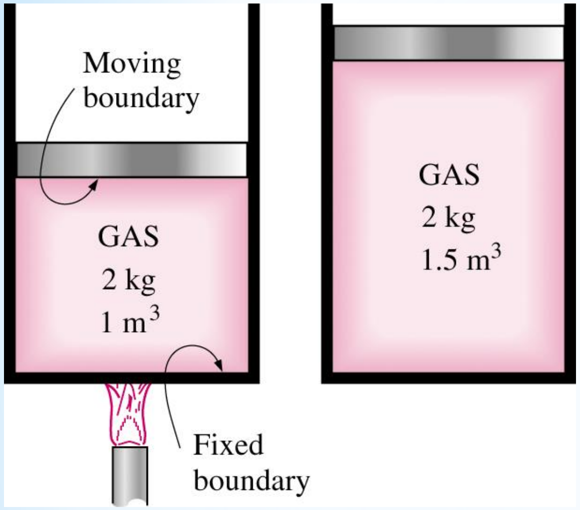
SOLUTION It is to be determined whether airflow over a bullet is one-, two-, or three-dimensional.

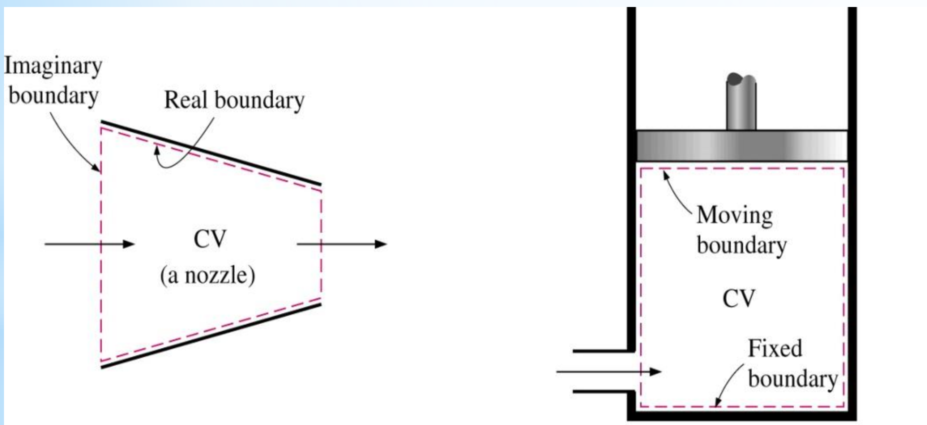
Assumptions There are no significant winds and the bullet is not spinning.

Analysis The bullet possesses an axis of symmetry and is therefore an axisymmetric body. The airflow upstream of the bullet is parallel to this axis, and we expect the time-averaged airflow to be rotationally symmetric about the axis—such flows are said to be axisymmetric. The velocity in this case varies with axial distance z and radial distance r, but not with angle θ. Therefore, the time-averaged airflow over the bullet is two-dimensional.

Discussion While the time-averaged airflow is axisymmetric, the *instantaneous* airflow is not, as illustrated in Fig. 1–19.

*System and Control Volume





(*a*) A control volume (CV) with real and imaginary boundaries

(*b*) A control volume (CV) with fixed and moving boundaries

- A system is defined as a quantity of matter or a region in space chosen for study.
- A closed system (known as a control mass) consists of a fixed amount of mass.
- An open system, or control volume, is a properly selected region in space. It usually encloses a device that involves mass flow such as a compressor, turbine, or nozzle.

*System and Control Volume

- *In general, *any arbitrary region in space* can be selected as a control volume. There are no concrete rules for the selection of control volumes, but the proper choice certainly makes the analysis much easier.
- *We'll discuss control volumes in more detail in Chapter 6.

*Dimensions and Units

Force Units

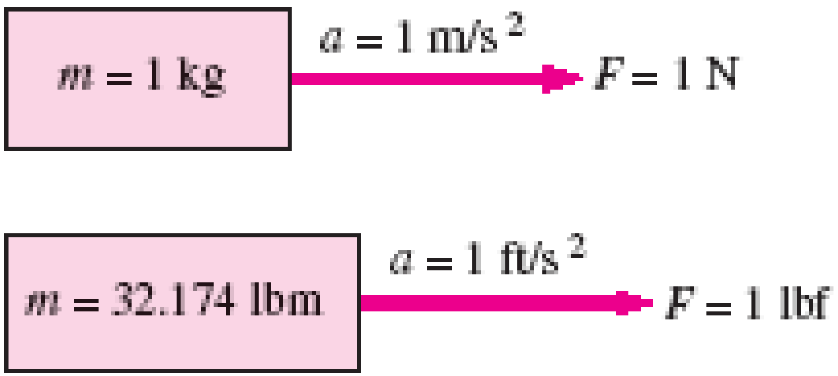


FIGURE 1–27
The definition of the force units.

We call a mass of *32.174 lbm 1 slug*

*Dimensions and Units

■**Weight** *W* is a *force*. It is the gravitational force applied to a body, and its magnitude is determined from Newton's second law,

$$W = mg \quad (\mathrm{N})$$

where *m* is the mass of the body, and *g* is the local gravitational acceleration (*g* is 9.807 m/s2 or 32.174 ft/s2 at sea level and 45° latitude).

■The weight of a unit volume of a substance is called the **specific weight** γ and is determined from $\gamma = \rho g$, where ρ is density.

*Dimensions and Units

■*Work*, which is a form of energy, can simply be defined as force times distance; therefore, it has the unit "newton-meter (N . m)," which is called a joule (J). That is,

$$1\ \mathrm{J} = 1\ \mathrm{N \cdot m}$$

■A more common unit for energy in SI is the kilojoule (1 kJ = 10^3 J). In the English system, the energy unit is the **Btu** (British thermal unit), which is defined as the energy required to raise the temperature of 1 lbm of water at 68°F by 1°F.

■In the metric system, the amount of energy needed to raise the temperature of 1 g of water at 14.5°C by 1°C is defined as 1 **calorie** (cal), and 1 cal = 4.1868 J. The magnitudes of the kilojoule and Btu are almost identical (1 Btu = 1.0551 kJ).

*Dimensions and Units

* **Dimensional homogeneity** is a valuable tool in checking for errors. Make sure every term in an equation has the same units.

EXAMPLE 1–2 Spotting Errors from Unit Inconsistencies

While solving a problem, a person ended up with the following equation at some stage:

$$E = 25 \text{ kJ} + 7 \text{ kJ/kg}$$

where E is the total energy and has the unit of kilojoules. Determine how to correct the error and discuss what may have caused it.

SOLUTION During an analysis, a relation with inconsistent units is obtained. A correction is to be found, and the probable cause of the error is to be determined.

Analysis The two terms on the right-hand side do not have the same units, and therefore they cannot be added to obtain the total energy. Multiplying the last term by mass will eliminate the kilograms in the denominator, and the whole equation will become dimensionally homogeneous; that is, every term in the equation will have the same unit.

Discussion Obviously this error was caused by forgetting to multiply the last term by mass at an earlier stage.

Dimensions and Units

- **Unity conversion ratios** are helpful in converting units. Use them.
- *All nonprimary units (**secondary units**) can be formed by combinations of primary units*. Force units, for example, can be expressed as

$$N = kg \frac{m}{s^2} \quad \text{and} \quad lbf = 32.174 \; lbm \frac{ft}{s^2}$$

They can also be expressed more conveniently as **unity conversion ratios** as

$$\frac{N}{kg \cdot m/s^2} = 1 \quad \text{and} \quad \frac{lbf}{32.174 \; lbm \cdot ft/s^2} = 1$$

EXAMPLE 1–4 **The Weight of One Pound-Mass**

Using unity conversion ratios, show that 1.00 lbm weighs 1.00 lbf on earth (Fig. 1–33).

Solution A mass of 1.00 lbm is subjected to standard earth gravity. Its weight in lbf is to be determined.

Assumptions Standard sea-level conditions are assumed.

Properties The gravitational constant is $g = 32.174 \text{ ft/s}^2$.

Analysis We apply Newton's second law to calculate the weight (force) that corresponds to the known mass and acceleration. The weight of any object is equal to its mass times the local value of gravitational acceleration. Thus,

$$W = mg = (1.00 \text{ lbm})(32.174 \text{ ft/s}^2)\left(\frac{1 \text{ lbf}}{32.174 \text{ lbm} \cdot \text{ft/s}^2}\right) = \mathbf{1.00 \text{ lbf}}$$

Discussion Mass is the same regardless of its location. However, on some other planet with a different value of gravitational acceleration, the weight of 1 lbm would differ from that calculated here.

* ▪ MATHEMATICAL MODELING OF ENGINEERING PROBLEMS

* An engineering device or process can be studied either
  * *experimentally* (testing and taking measurements)

    Advantage : deal with the actual physical system, and the desired quantity is determined by measurement, within the limits of experimental error.

    Drawback: approach is expensive, time-consuming, and often impractical. Besides, the system we are studying may not even exist.
  * *analytically* (by analysis or calculations).

    Advantage : fast and inexpensive

    Drawback: the results obtained are subject to the accuracy of the assumptions, approximations, and idealizations made in the analysis.
* In engineering studies, often a good compromise is reached by reducing the choices to just a few by analysis, and then verifying the findings experimentally.

* MATHEMATICAL MODELING OF ENGINEERING PROBLEMS

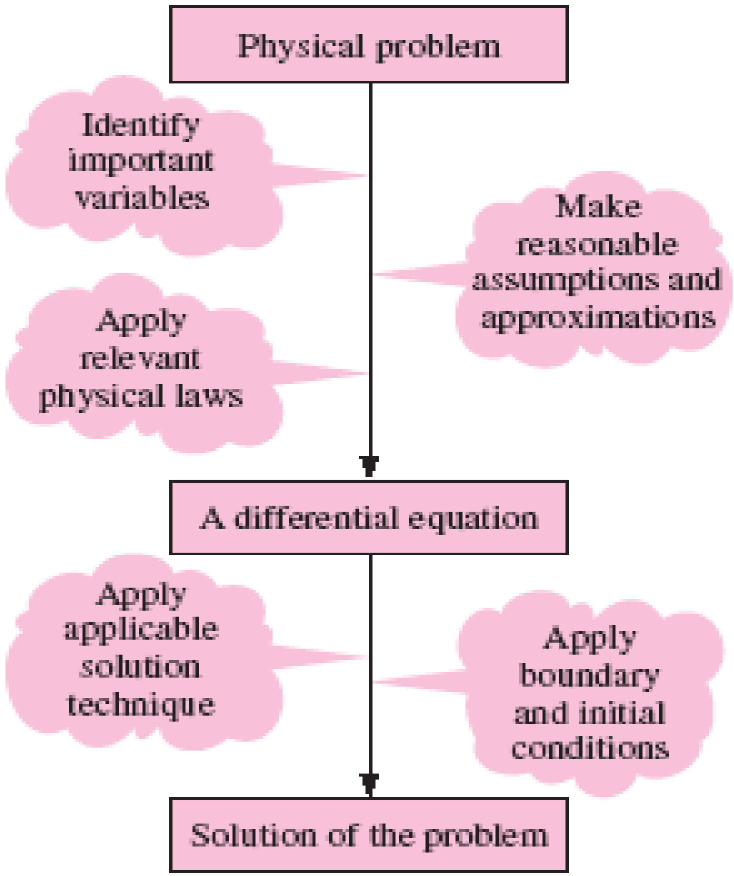


FIGURE 1–35
Mathematical modeling of physical problems.

* MATHEMATICAL MODELING OF ENGINEERING PROBLEMS

* The study of physical phenomena involves two important steps.
  * In the first step, all the variables that affect the phenomena are identified, reasonable assumptions and approximations are made, and the interdependence of these variables is studied. The relevant physical laws and principles are invoked, and the problem is formulated mathematically. The equation itselfis very instructive as it shows the degree of dependence of some variables on others, and the relative importance of various terms.
  * In the second step the problem is solved using an appropriate approach, and the results are interpreted.

*PROBLEM-SOLVING TECHNIQUE

* using a step-by-step approach, an engineer can reduce the solution of a complicated problem into the solution of a series of simple problems.

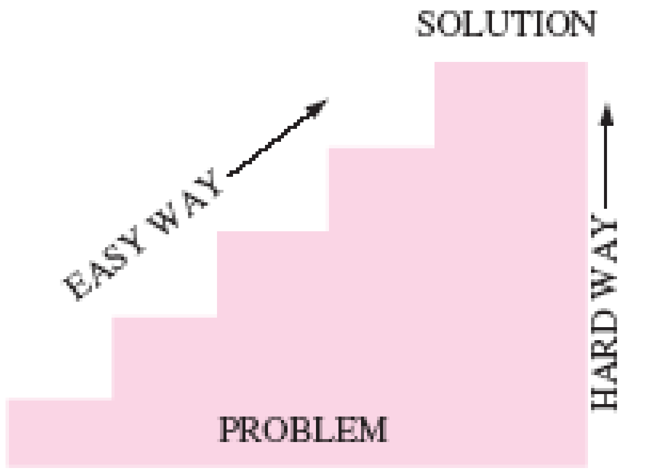


FIGURE 1–36
A step-by-step approach can greatly simplify problem solving.

PROBLEM-SOLVING TECHNIQUE

- Step 1: Problem Statement
- Step 2: Schematic
- Step 3: Assumptions and Approximations
- Step 4: Physical Laws
- Step 5: Properties
- Step 6: Calculations
- Step 7: Reasoning, Verification, and Discussion

* Reasoning, Verification, and Discussion

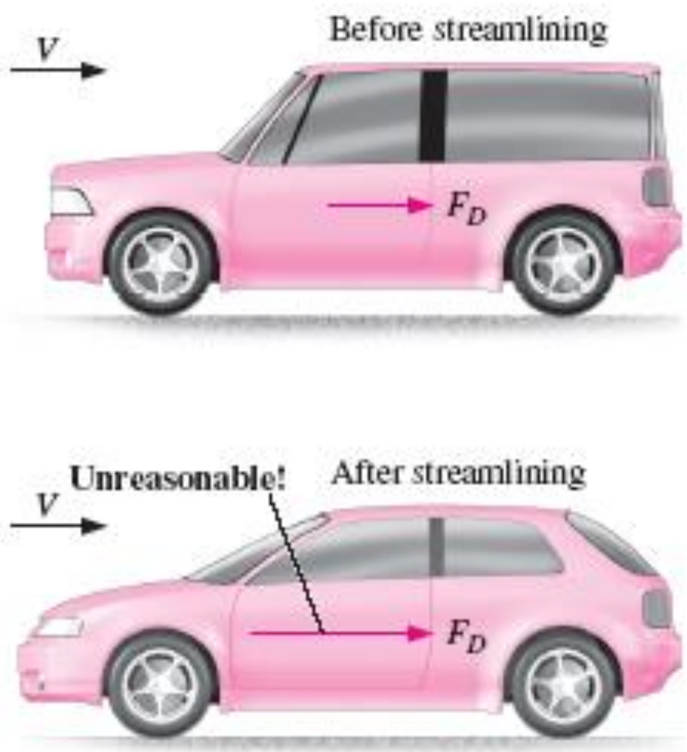


FIGURE 1–38
The results obtained from an engineering analysis must be checked for reasonableness.

* ENGINEERING SOFTWARE PACKAGES

* **Engineering Equation Solver (EES)** is a program that solves systems of linear or nonlinear algebraic or differential equations numerically.
* **FLUENT** is a computational fluid dynamics (CFD) code widely used for flow-modeling applications.

Please refer to section 1-9 in the text book.

* Accuracy, Precision, and Significant Digits

Engineers must be aware of three principals that govern the proper use of numbers.

1. ***Accuracy error :*** Value of one reading minus the true value. Closeness of the average reading to the true value. Generally associated with repeatable, fixed errors.
2. ***Precision error :*** Value of one reading minus the average of readings. Is a measure of the fineness of resolution and repeatability of the instrument. Generally associated with random errors.
3. ***Significant digits :*** Digits that are relevant and meaningful. When performing calculations, the final result is only as precise as the least precise parameter in the problem. When the number of significant digits is unknown, the accepted standard is 3. Use 3 in all homework and exams.

* Accuracy, Precision, and Significant Digits

* A measurement or calculation can be very precise without being very accurate, and vice versa. For example, suppose the true value of wind speed is 25.00 m/s. Two anemometers A and B take five wind speed readings each: *Anemometer A*: 25.50, 25.69, 25.52, 25.58, and 25.61 m/s. Average of all readings = 25.58 m/s. *Anemometer B*: 26.3, 24.5, 23.9, 26.8, and 23.6 m/s. Average of all readings = 25.02 m/s.

Accuracy, Precision, and Significant Digits

In engineering calculations, the supplied information is not known to more than a certain number of significant digits, usually three digits.

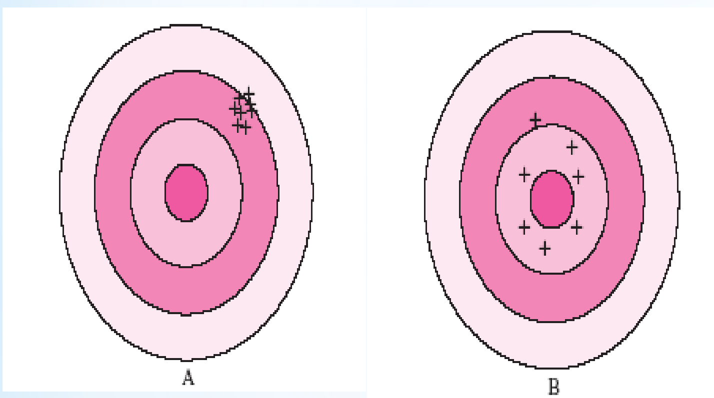


FIGURE 1–40

Illustration of accuracy versus precision. Shooter A is more precise, but less accurate, while shooter B is more accurate, but less precise.

Given: Volume: $V = 3.75$ L

Density: $\rho = 0.845$ kg/L

(3 significant digits)

Also, $3.75 \times 0.845 = 3.16875$

Find: Mass: $m = \rho V = 3.16875$ kg

Rounding to 3 significant digits:

$m = 3.17$ kg

EXAMPLE 1–6 Significant Digits and Volume Flow Rate

Jennifer is conducting an experiment that uses cooling water from a garden hose. In order to calculate the volume flow rate of water through the hose, she times how long it takes to fill a container (Fig. 1–42). The volume of water collected is $V = 1.1$ gal in time period $\Delta t = 45.62$ s, as measured with a stopwatch. Calculate the volume flow rate of water through the hose in units of cubic meters per minute.

SOLUTION Volume flow rate is to be determined from measurements of volume and time period.

Assumptions 1 Jennifer recorded her measurements properly, such that the volume measurement is precise to two significant digits while the time period is precise to four significant digits. 2 No water is lost due to splashing out of the container.

Analysis Volume flow rate $\dot{V}$ is volume displaced per unit time and is expressed as

Volume flow rate:

$$\dot{V} = \frac{\Delta V}{\Delta t}$$

Substituting the measured values, the volume flow rate is determined to be

$$\dot{V} = \frac{1.1 \text{ gal}}{45.62 \text{ s}} \left(\frac{3.785 \times 10^{-3} \text{ m}^3}{1 \text{ gal}} \right) \left(\frac{60 \text{ s}}{1 \text{ min}} \right) = \mathbf{5.5 \times 10^{-3} \text{ m}^3/\text{min}}$$

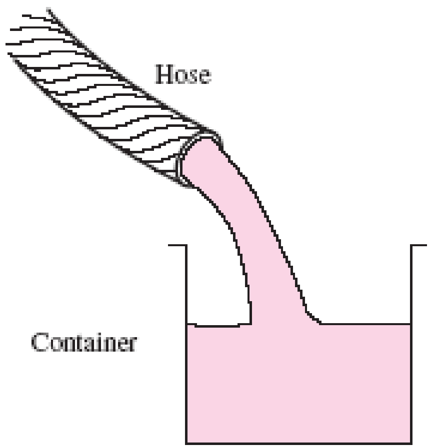


FIGURE 1–42
Schematic for Example 1–6 for the measurement of volume flow rate.

Discussion The final result is listed to two significant digits since we cannot be confident of any more precision than that. If this were an intermediate step in subsequent calculations, a few extra digits would be carried along to avoid accumulated round-off error. In such a case, the volume flow rate would be written as $\dot{V} = 5.4759 \times 10^{-3}$ m³/min. Based on the given information, we cannot say anything about the *accuracy* of our result, since we have no information about systematic errors in either the volume measurement or the time measurement.

Also keep in mind that good precision does not guarantee good accuracy. For example, if the batteries in the stopwatch were weak, its accuracy could be quite poor, yet the readout would still be displayed to four significant digits of precision.

In common practice, precision is often associated with *resolution*, which is a measure of how finely the instrument can report the measurement. For example, a digital voltmeter with five digits on its display is said to be more

Accuracy, Precision, and Significant Digits

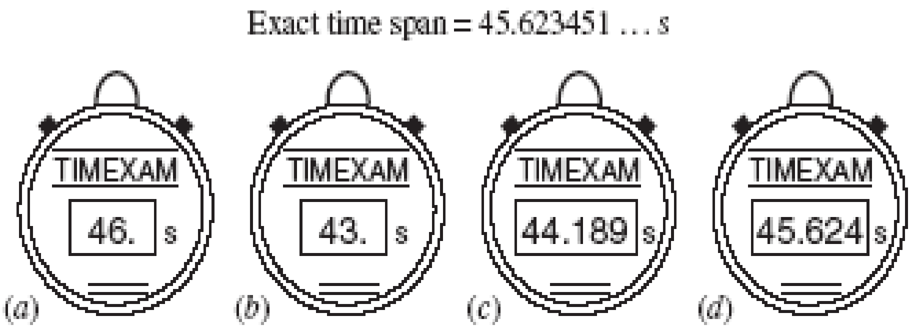


FIGURE 1–43
An instrument with many digits of resolution (stopwatch *c*) may be less accurate than an instrument with few digits of resolution (stopwatch *a*). What can you say about stopwatches *b* and *d*?

precise than a digital voltmeter with only three digits. However, the number of displayed digits has nothing to do with the overall *accuracy* of the measurement. An instrument can be very precise without being very accurate when there are significant bias errors. Likewise, an instrument with very few displayed digits can be more accurate than one with many digits (Fig. 1–43).

*Summary

In this chapter some basic concepts of fluid mechanics are introduced and discussed.

* A substance in the liquid or gas phase is referred to as a *fluid*. *Fluid mechanics* is the science that deals with the behavior of fluids at rest or in motion and the interaction of fluids with solids or other fluids at the boundaries.
* The flow of an unbounded fluid over a surface is *external flow*, and the flow in a pipe or duct is *internal flow* if the fluid is completely bounded by solid surfaces.
* A fluid flow is classified as being *compressible* or *incompressible*, depending on the density variation of the fluid during flow. The densities of liquids are essentially constant, and thus the flow of liquids is typically incompressible.
* The term *steady* implies *no change with time*. The opposite of steady is *unsteady*, or *transient*.
* The term *uniform* implies *no change with location* over a specified region.
* A flow is said to be *one-dimensional* when the velocity changes in one dimension only.

*Summary

* A fluid in direct contact with a solid surface sticks to the surface and there is no slip. This is known as the *no-slip condition*, which leads to the formation of *boundary layers* along solid surfaces.
* A system of fixed mass is called a *closed system*, and a system that involves mass transfer across its boundaries is called an *open system* or *control volume*. A large number of engineering problems involve mass flow in and out of a system and are therefore modeled as *control volumes*.
* In engineering calculations, it is important to pay particular attention to the units of the quantities to avoid errors caused by inconsistent units, and to follow a systematic approach.
* It is also important to recognize that the information given is not known to more than a certain number of significant digits, and the results obtained cannot possibly be accurate to more significant digits.

  The information given on dimensions and units; problem-solving technique; and accuracy, precision, and significant digits will be used throughout the entire text.

2.2.1 Absolute system of units

Y. NAKAYAMA
Former Professor, Tokai University, Japan

MKS system of units

This is the system of units where the metre (m) is used for the unit of length, kilogram (kg) for the unit of mass, and second (s) for the unit of time as the base units.

CGS system of units

This is the system of units where the centimetre (cm) is used for length, gram (g) for mass, and second (s) for time as the base units.

International system of units (SI)

SI, the abbreviation of La Système International d'Unités, is the system developed from the MKS system of units. It is a consistent and reasonable system of units which makes it a rule to adopt only one unit for each of the various quantities used in such fields as science, education and industry.

There are seven fundamental SI units, namely: metre (m) for length, kilogram (kg) for mass, second (s) for time, ampere (A) for electric current, kelvin (K) for thermodynamic temperature, mole (mol) for mass quantity and candela (cd) for intensity of light. Derived units consist of these units.

Y. NAKAYAMA
Former Professor, Tokai University, Japan

Table 2.1 Dimensions and units

Quantity	Absolute system of units			
	α	β	γ	Units
Length	1	0	0	m
Mass	0	1	0	kg
Time	0	0	1	s
Velocity	1	0	−1	m/s
Acceleration	1	0	−2	m/s^2
Density	−3	1	0	kg/m^3
Force	1	1	−2	$N = kg\,m/s^2$
Pressure, stress	−1	1	−2	$Pa = N/m^2$
Energy, work	2	1	−2	J
Viscosity	−1	1	−1	Pa s
Kinematic viscosity	2	0	−1	m^2/s

Dimensions and Units

- Any physical quantity can be characterized by **dimensions.**
- The magnitudes assigned to dimensions are called **units.**
- Primary dimensions (or fundamental dimensions) include: mass *m*, length *L*, time *t*, and temperature *T*, *etc*.

TABLE 1–1

The seven fundamental (or primary) dimensions and their units in SI

Dimension	Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Temperature	kelvin (K)
Electric current	ampere (A)
Amount of light	candela (cd)
Amount of matter	mole (mol)

By *General Conference of Weights and Measures*

*Dimensions and Units

- Secondary dimensions (derived dimensions) can be expressed in terms of primary dimensions and include: velocity *V*, energy *E*, and volume *V*.
- Unit systems include English system and the metric SI (International System). We'll use both.

* Dimensions and Units

Based on the notational scheme introduced in 1967,

* The degree symbol was officially dropped from the absolute temperature unit,
* All unit names were to be written without capitalization even if they were derived from proper names (Table 1–1).
* However, the abbreviation of a unit was to be capitalized if the unit was derived from a proper name. For example, the SI unit of force, which is named after Sir Isaac Newton (1647–1723), is *newton* (not Newton), and it is abbreviated as N.
* Also, the full name of a unit may be pluralized, but its abbreviation cannot. For example, the length of an object can be 5 m or 5 meters, *not* 5 ms or 5 meter.
* Finally, no period is to be used in unit abbreviations unless they appear at the end of a sentence. For example, the proper abbreviation of meter is m (not m.).

*Dimensions and Units

Some SI and English Units

* In SI, the units of mass, length, and time are the kilogram (kg), meter (m), and second (s), respectively. The respective units in the English system are the pound-mass (lbm), foot (ft), and second (s).

$$1 \text{ lbm} = 0.45359 \text{ kg}$$

$$1 \text{ ft} = 0.3048 \text{ m}$$

Y. NAKAYAMA
Former Professor, Tokai University, Japan

2.3 Density, specific gravity and specific volume

The mass per unit volume of material is called the density, which is generally expressed by the symbol ρ. The density of a gas changes according to the pressure, but that of a liquid may be considered unchangeable in general. The units of density are kg/m^3 (SI). The density of water at 4°C and 1 atm (101 325 Pa, standard atmospheric pressure; see Section 3.1.1) is $1000\,kg/m^3$.

The ratio of the density of a material ρ to the density of water ρ_w is called the specific gravity, which is expressed by the symbol s:

$$s = \rho/\rho_w \tag{2.2}$$

The reciprocal of density, i.e. the volume per unit mass, is called the specific volume, which is generally expressed by the symbol v:

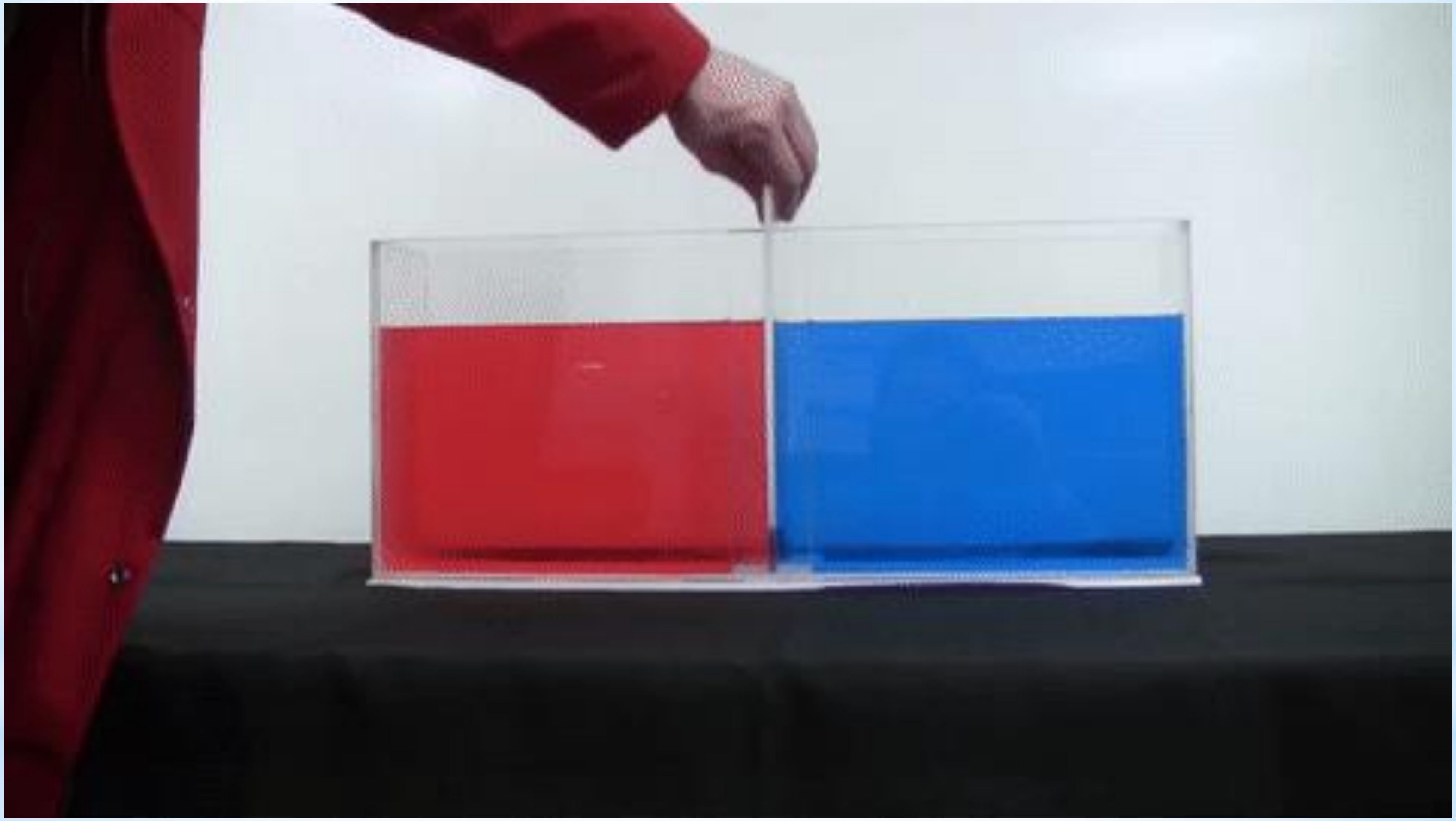
$$v = 1/\rho \quad (m^3/kg) \tag{2.3}$$

Values for the density ρ of water and air under standard atmospheric pressure are given in Table 2.2.

Table 2.2 Density of water and air (standard atmospheric pressure)

Temperature (°C)		**0**	**10**	**15**	**20**	**40**	**60**	**80**	**100**
ρ (kg/m^3)	Water	999.8	999.7	999.1	998.2	992.2	983.2	971.8	958.4
	Air	1.293	1.247	1.226	1.205	1.128	1.060	1.000	0.9464

Y. NAKAYAMA
Former Professor, Tokai University, Japan



FLUIDS MAY HAVE SAME VISCOSITY BUT DIFFERENT DENSITY

Y. NAKAYAMA
Former Professor, Tokai University, Japan

2.5 Surface tension

The surface of a liquid is apt to shrink, and its free surface is in such a state where each section pulls another as if an elastic film is being stretched. The tensile strength per unit length of assumed section on the free surface is called the surface tension. Surface tensions of various kinds of liquid are given in Table 2.4.

As shown in Fig. 2.5, a dewdrop appearing on a plant leaf is spherical in shape. This is also because of the tendency to shrink due to surface tension. Consequently its internal pressure is higher than its peripheral pressure. Putting d as the diameter of the liquid drop, T as the surface tension, and p as the increase in internal pressure, the following equation is obtained owing to the balance of forces as shown in Fig. 2.6:

$$\pi d T = \frac{\pi d^2}{4} \Delta p$$

or

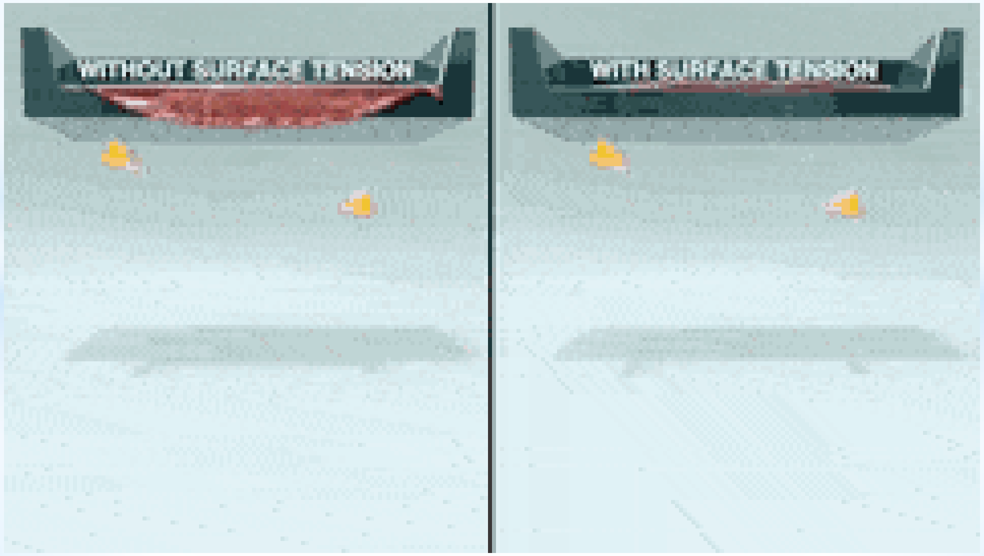
$$\Delta p = 4T/d \qquad (2.7)$$

The same applies to the case of small bubbles in a liquid.

Table 2.4 Surface tension of liquid (20°C)

Liquid	Surface liquid	N/m
Water	Air	0.0728
Mercury	Air	0.476
Mercury	Water	0.373
Methyl alcohol	Air	0.023

Y. NAKAYAMA
Former Professor, Tokai University, Japan



Y. NAKAYAMA
Former Professor, Tokai University, Japan

or

$$h = \frac{4T\cos\theta}{\rho g d} \tag{2.8}$$

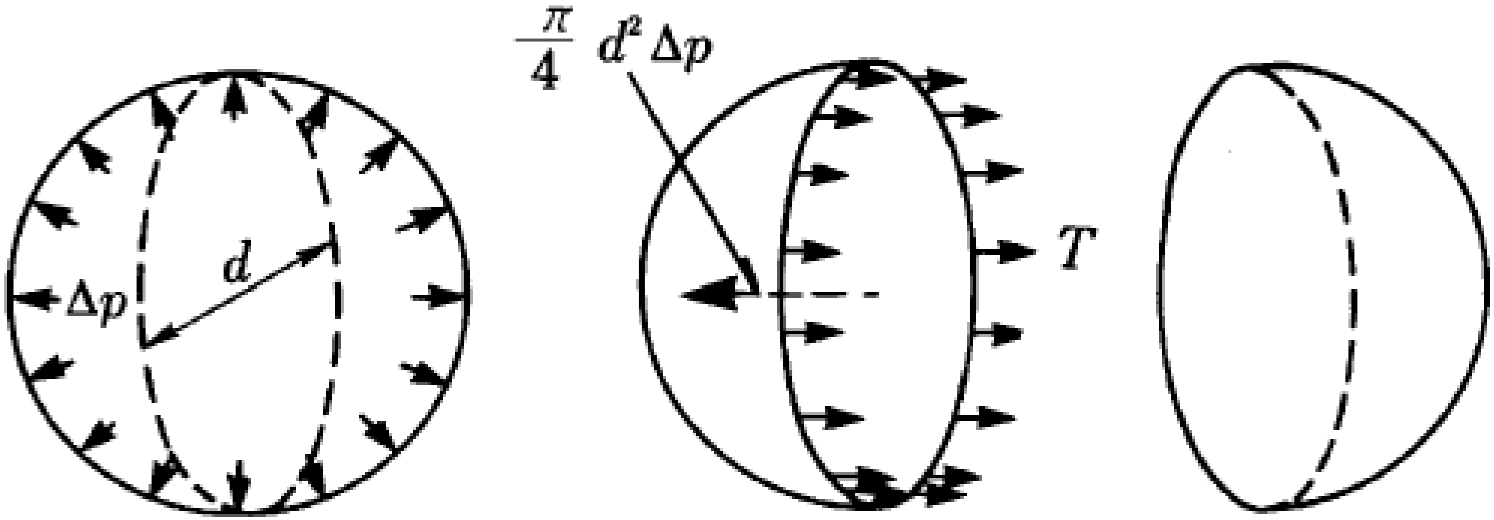


Fig. 2.6 Balance between the pressure increase within a liquid drop and the surface tension

2.6 Compressibility

Y. NAKAYAMA
Former Professor, Tokai University, Japan

As shown in Fig. 2.9, assume that fluid of volume V at pressure p decreased its volume by ΔV due to the further increase in pressure by Δp. In this case, since the cubic dilatation of the fluid is $\Delta V/V$, the bulk modulus K is expressed by the following equation:

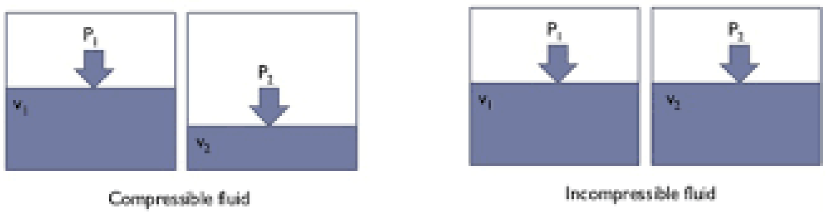
$$K = \frac{\Delta p}{\Delta V/V} = -V\frac{dp}{dV} \tag{2.10}$$

Its reciprocal β

$$\beta = 1/K \tag{2.11}$$

is called the compressibility, whose value directly indicates how compressible the fluid is. For water of normal temperature/pressure $K = 2.06 \times 10^9$ Pa, and for air $K = 1.4 \times 10^5$ Pa assuming adiabatic change. In the case of water, $\beta = 4.85 \times 10^{-10}$ 1/Pa, and shrinks only by approximately 0.005% even if the atmospheric pressure is increased by 1 atm.

Compressible and Incompressible flows

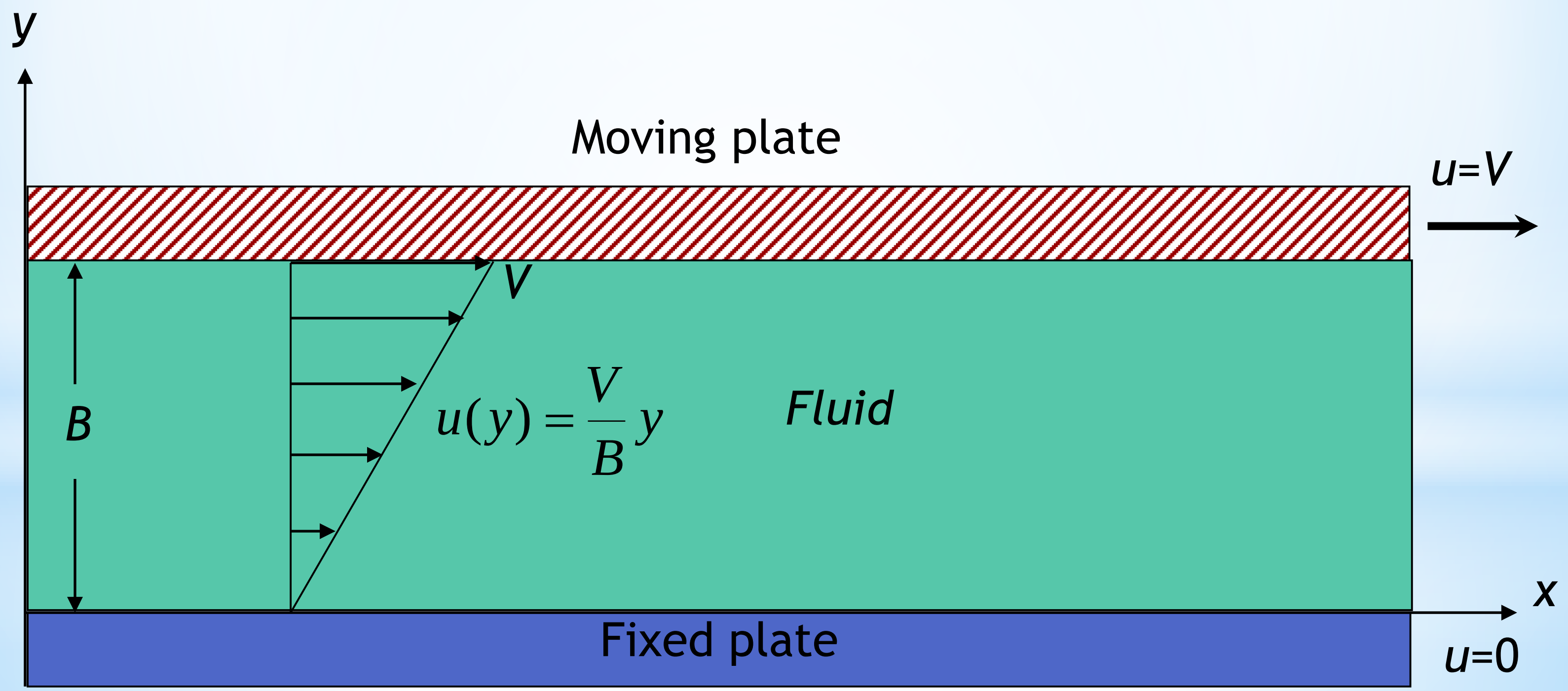


*Some Simple Flows

- Flow between a fixed and a moving plate

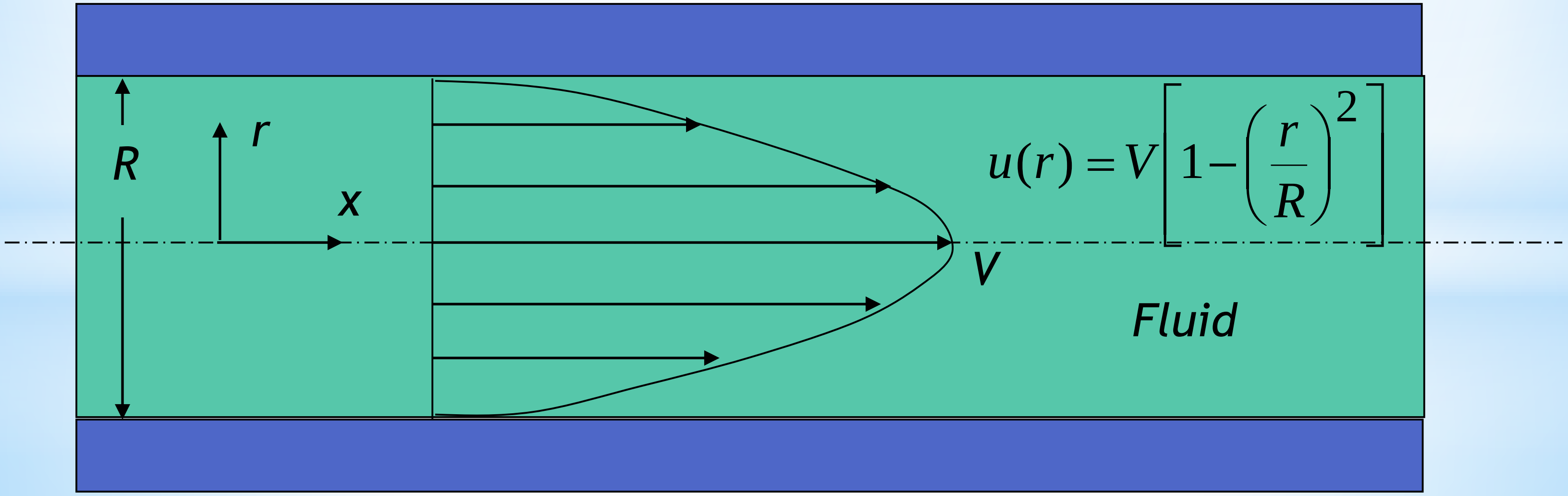
  Fluid in contact with the plate has the same velocity as the plate

  u = *x*-direction component of velocity



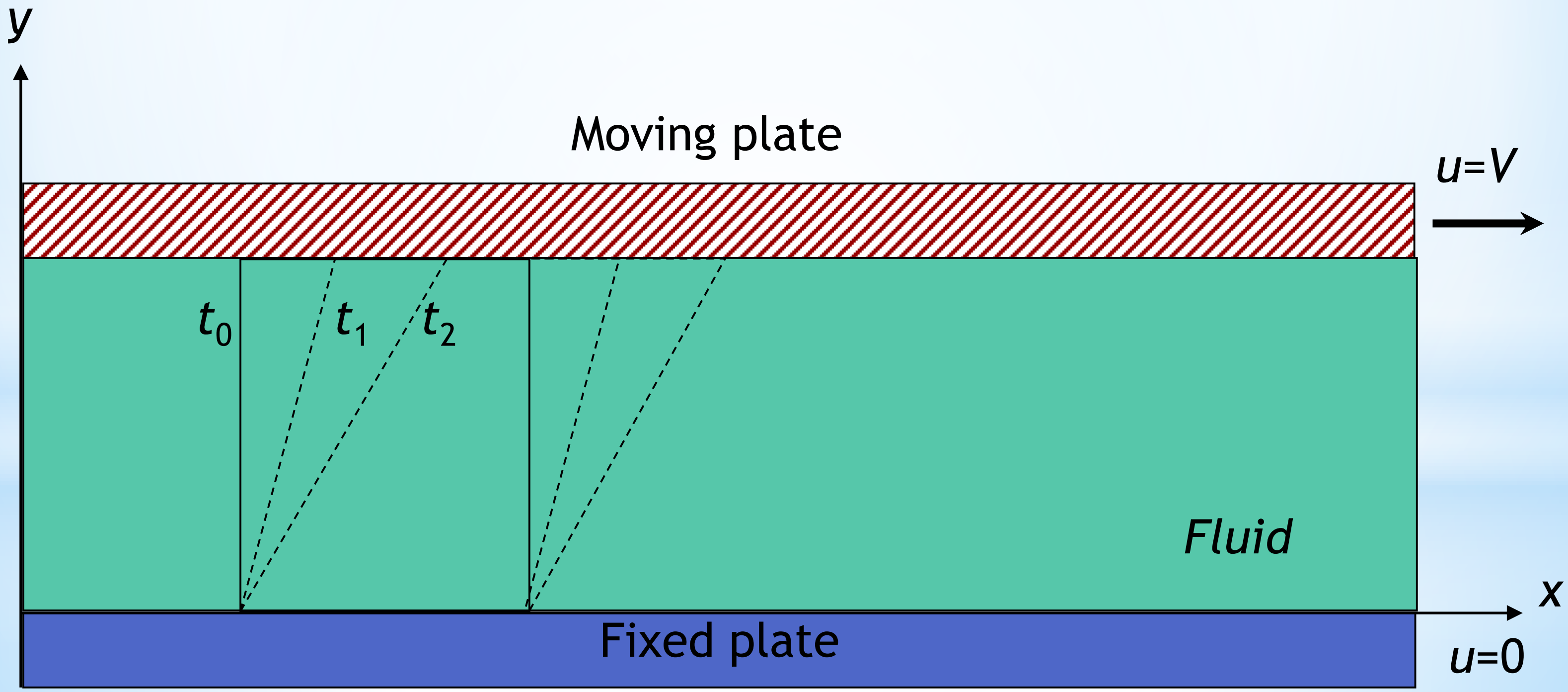
Some Simple Flows

- Flow through a long, straight pipe
  - Fluid in contact with the pipe wall has the same velocity as the wall
  - u = x-direction component of velocity



Fluid Deformation

- Flow between a fixed and a moving plate
- Force causes plate to move with velocity V and the fluid deforms continuously.

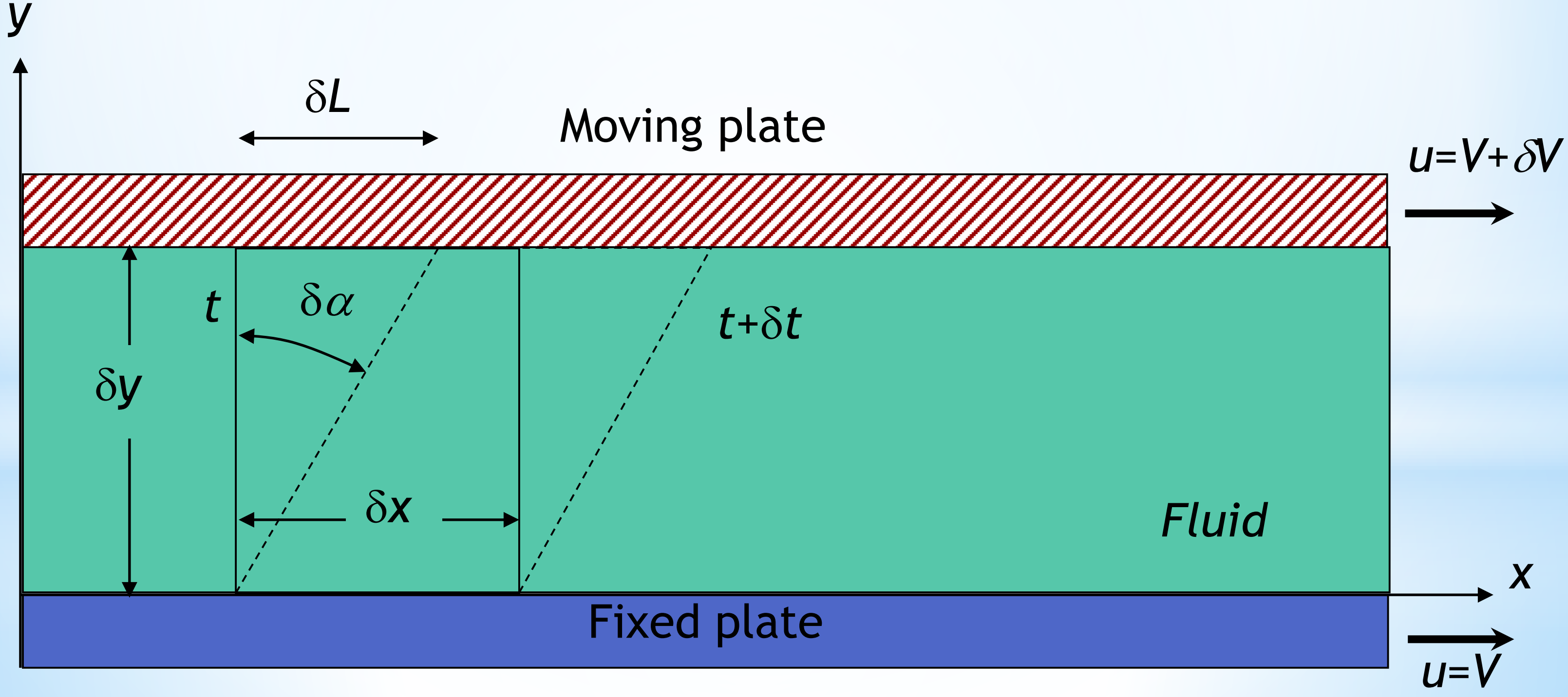


*Fluid Deformation

Shear stress on the plate is proportional to deformation rate of the fluid

$$\tau \propto \frac{\delta\alpha}{\delta t}$$

$$\delta\alpha = \frac{\delta L}{\delta y} \qquad \delta t = \frac{\delta L}{\delta V} \qquad \frac{\delta\alpha}{\delta t} = \frac{\delta V}{\delta y} \qquad \tau \propto \frac{\delta V}{\delta y}$$

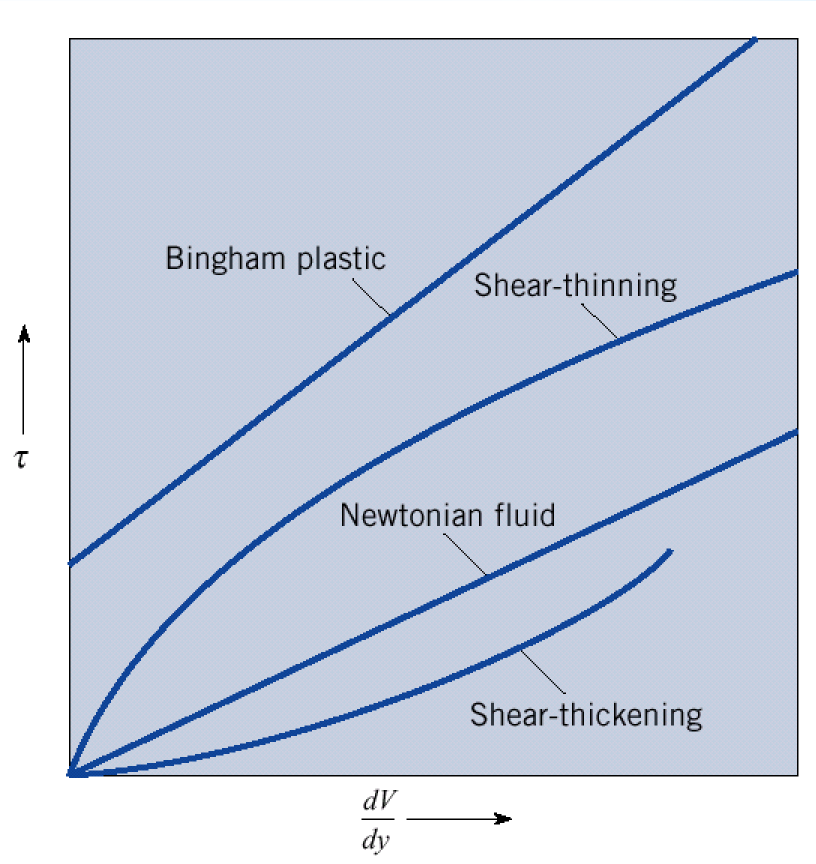


*Shear in Different Fluids

* Shear-stress relations for different types of fluids
* Newtonian fluids: linear relationship
* Slope of line (coefficient of proportionality) is "viscosity"

$$\tau \propto \frac{dV}{dy}$$

$$\tau = \mu \frac{dV}{dy}$$



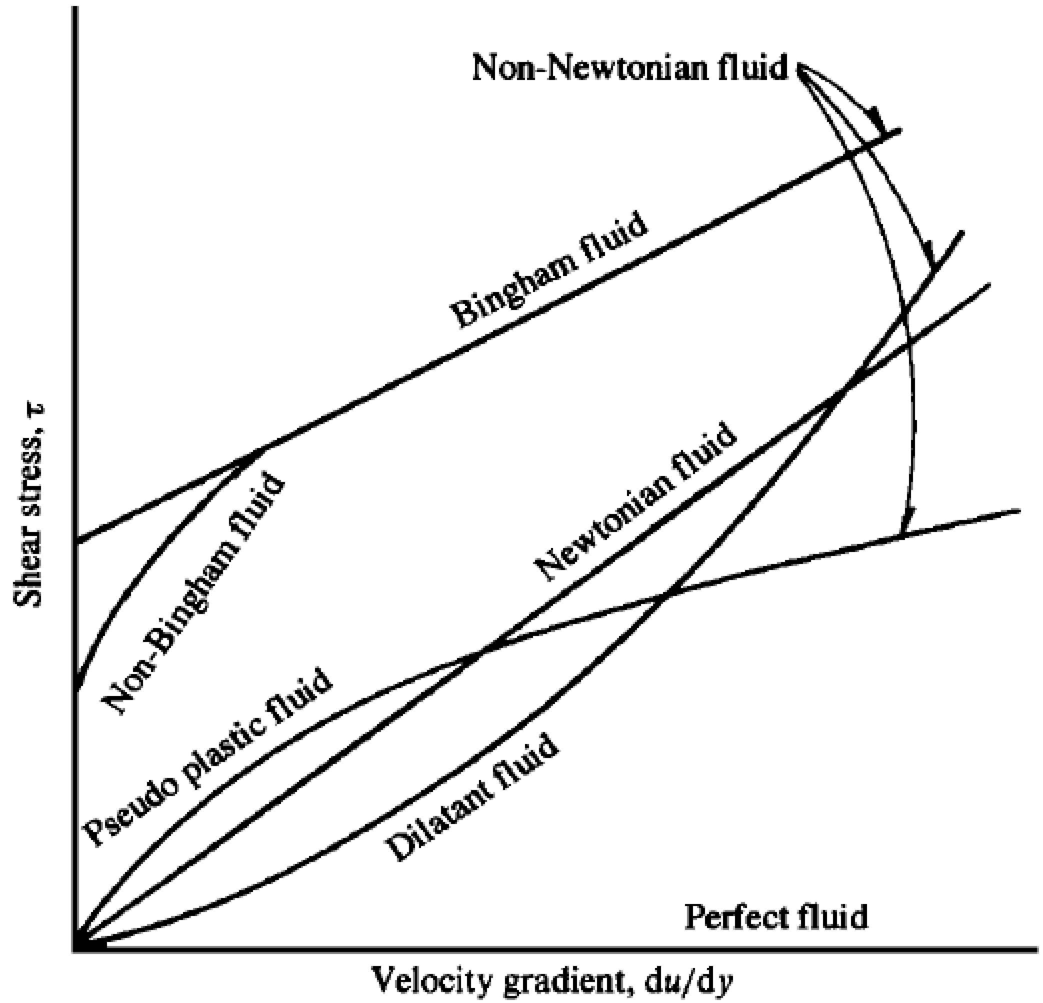
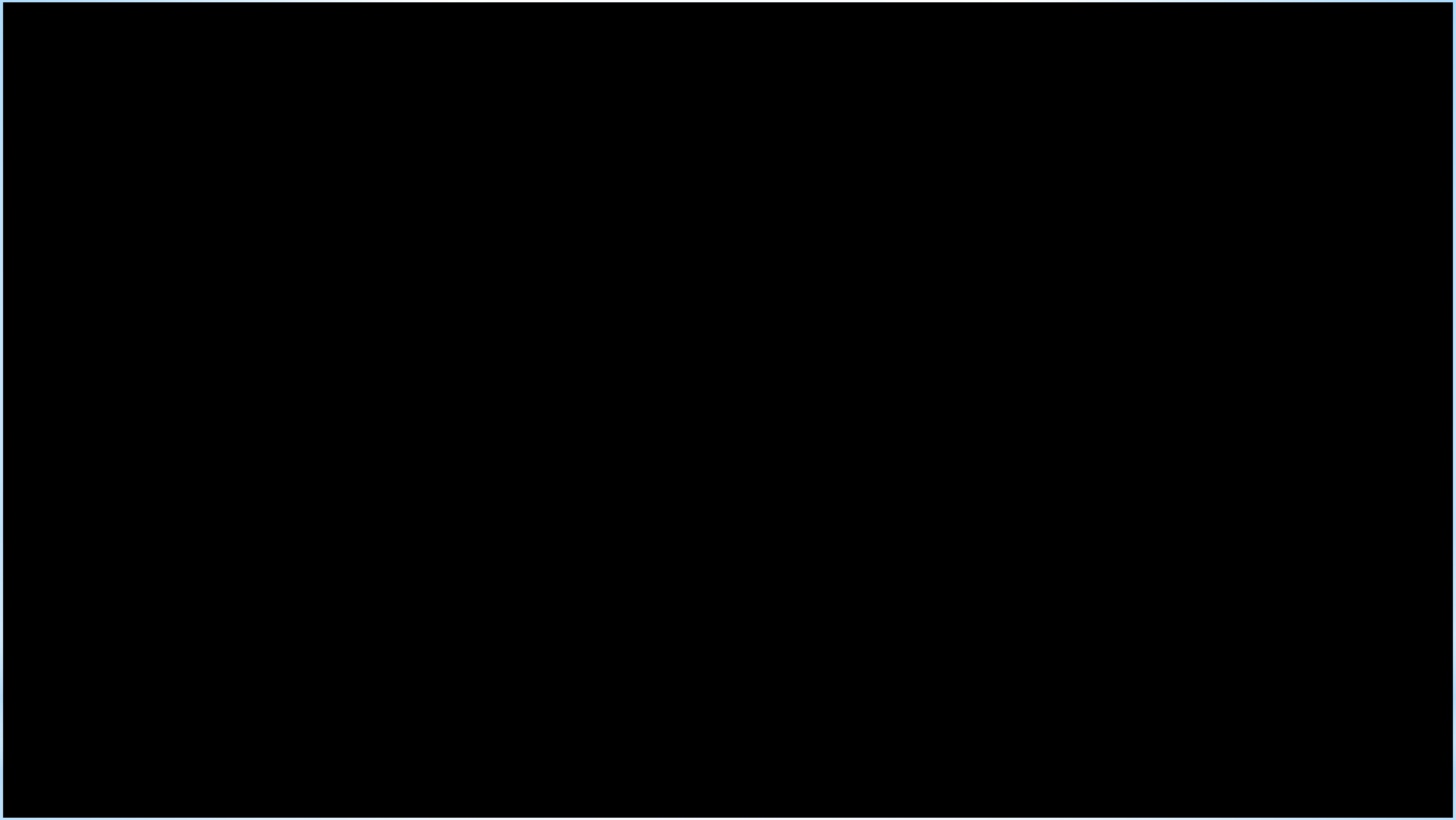
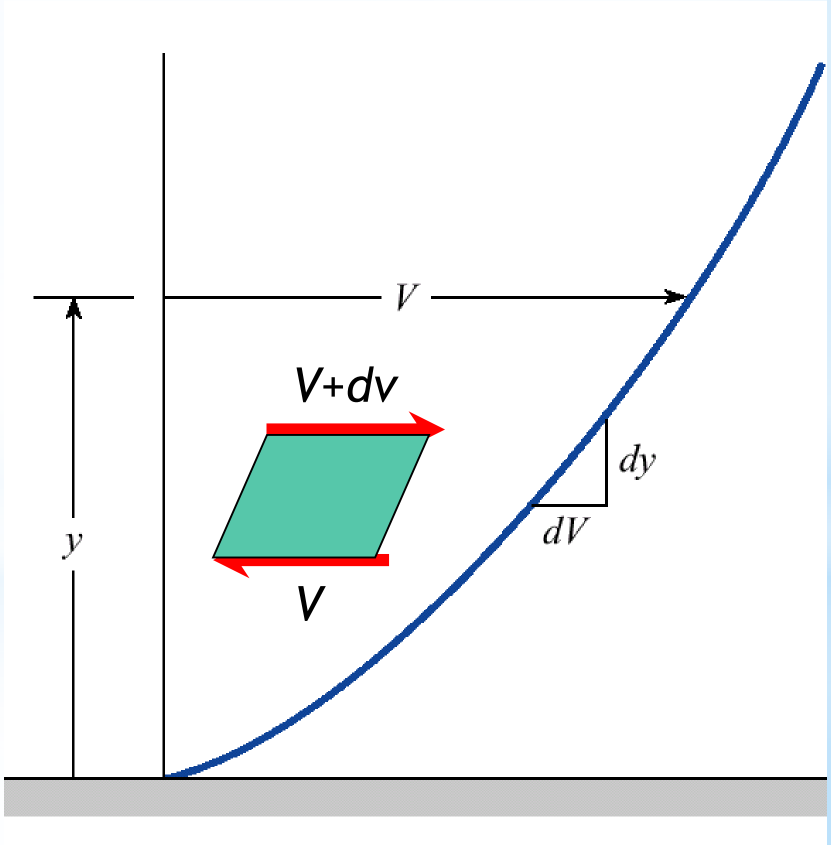


Fig. 2.4 Rheological diagram



*Viscosity

* Newton's Law of Viscosity $\tau = \mu \frac{dV}{dy}$
* Viscosity $\mu = \frac{\tau}{dV / dy}$
* Units $\frac{N / m^2}{m / s / m} = \frac{N \cdot s}{m^2}$
* Water (@ 20^{o}C)
  * μ = 1x10^{-3} *N-s/m*^{2}
* Air (@ 20^{o}C)
  * μ = 1.8x10^{-5} *N-s/m*^{2}
* Kinematic viscosity $\nu = \frac{\mu}{\rho}$



*Flow between 2 plates

Force is same on top and bottom

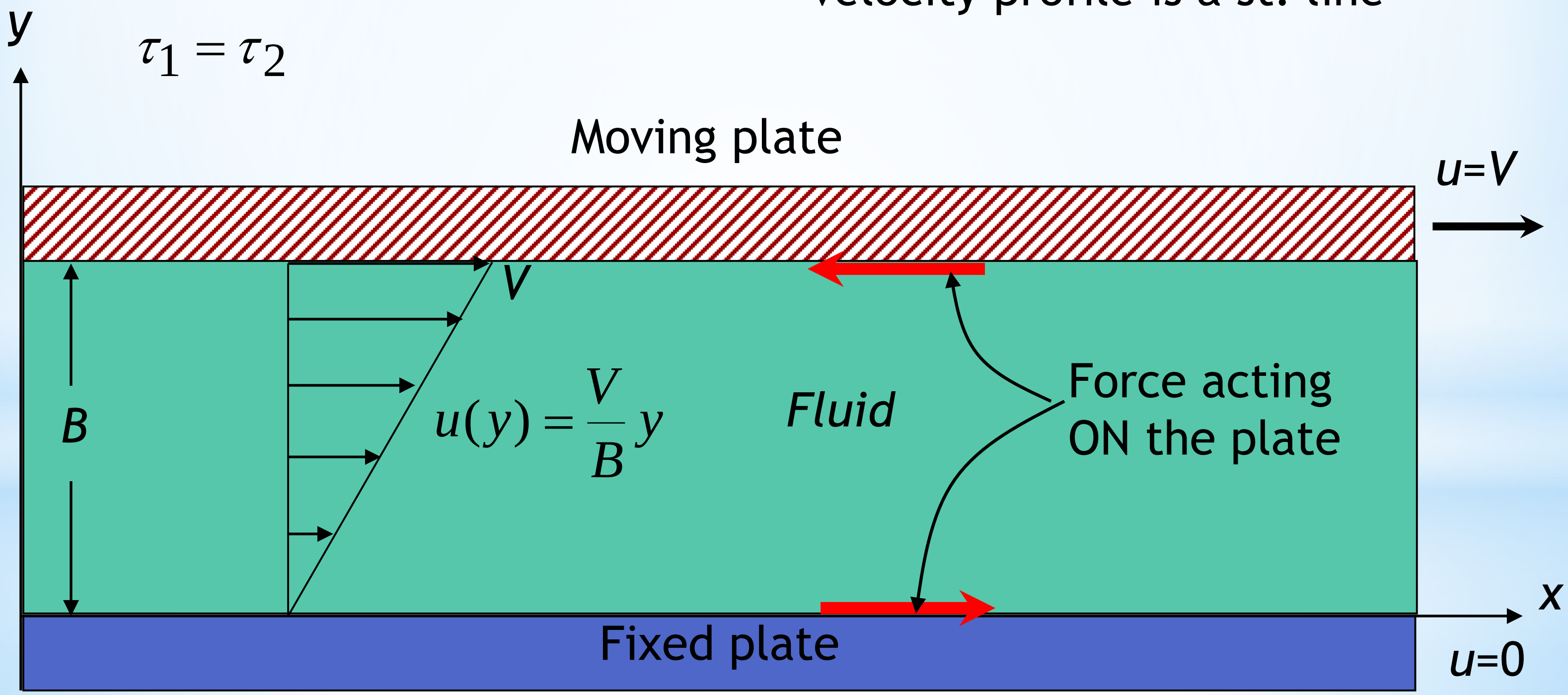
$$\tau_1 = \mu \left.\frac{du}{dy}\right|_1 = \mu \left.\frac{du}{dy}\right|_2 = \tau_2$$

$$F_1 = \tau_1 A_1 = \tau_2 A_2 = F_2$$

$$A_1 = A_2$$

$$\tau_1 = \tau_2$$

Thus, slope of velocity profile is constant and velocity profile is a st. line



*Flow between 2 plates

Shear stress anywhere between plates

$$\tau = \mu \frac{du}{dy} = \mu \frac{V}{B}$$

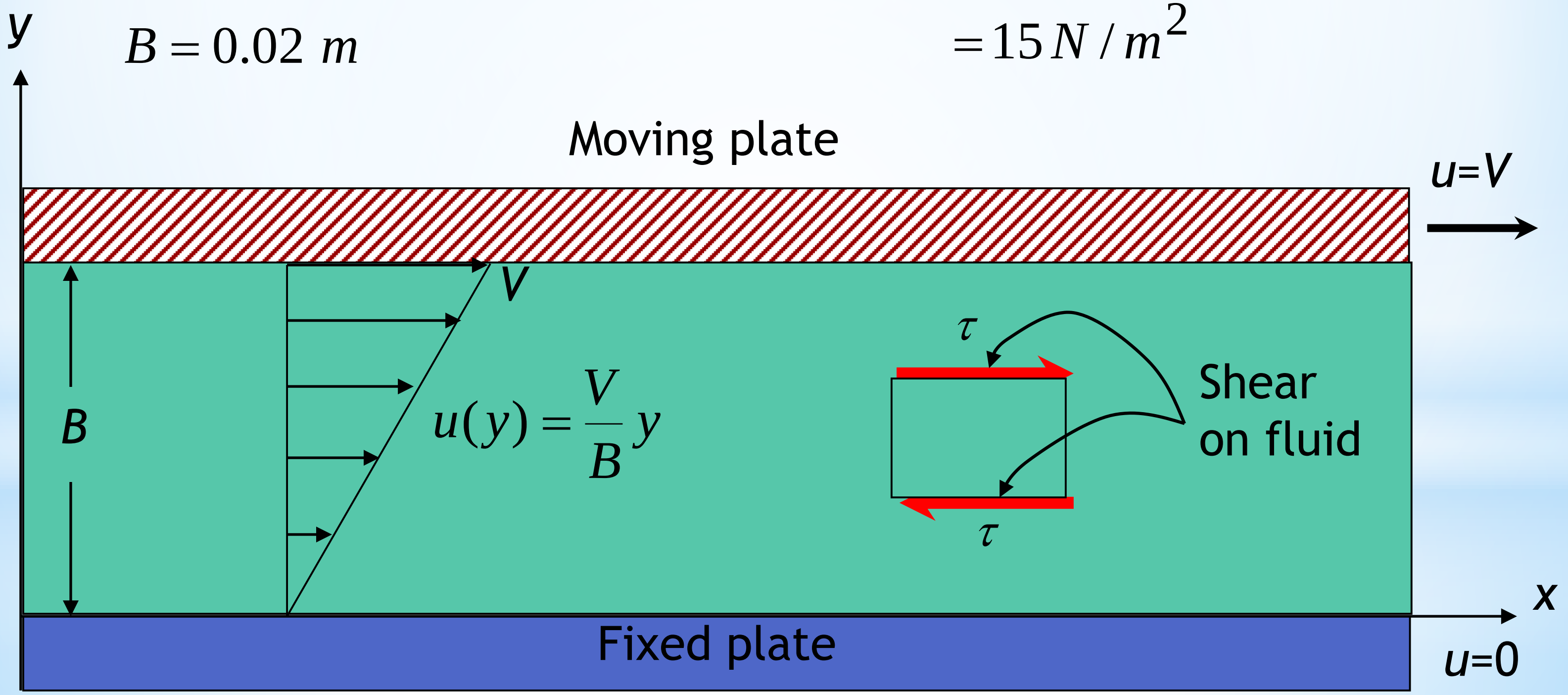
$$\mu = 0.1\ N \cdot s/m^2\ (SAE\,30 @ 38^o C)$$

$$V = 3\ m/s$$

$$B = 0.02\ m$$

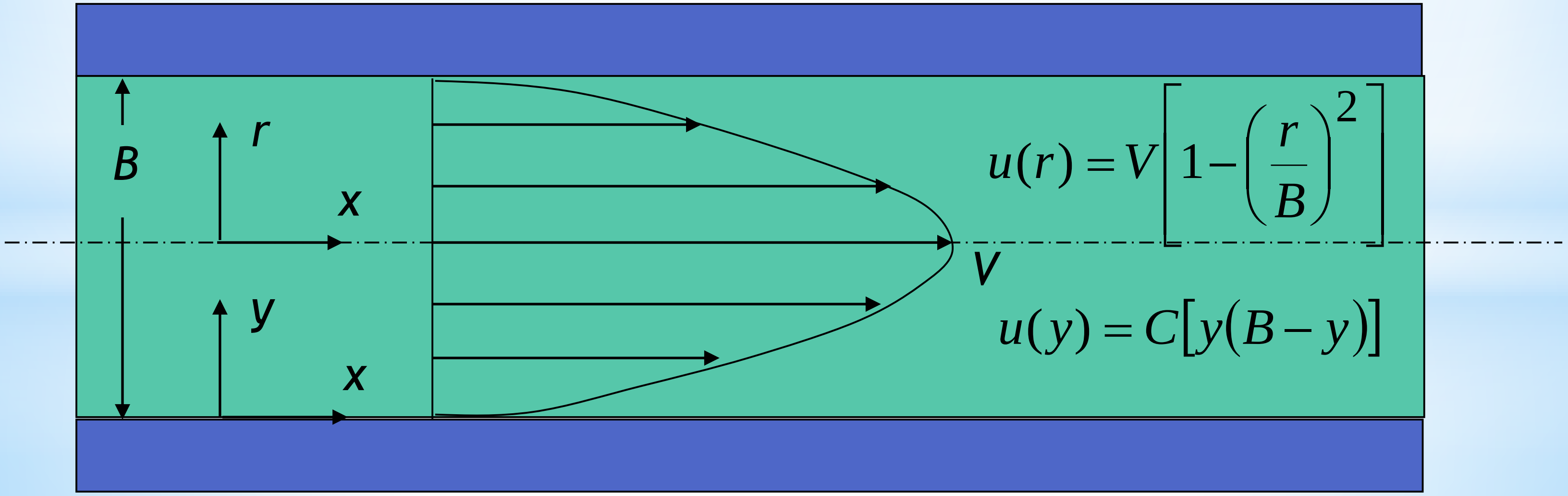
$$\tau = (0.1\,N \cdot s/m^2)(\frac{3\,m/s}{0.02\,m})$$

$$= 15\,N/m^2$$



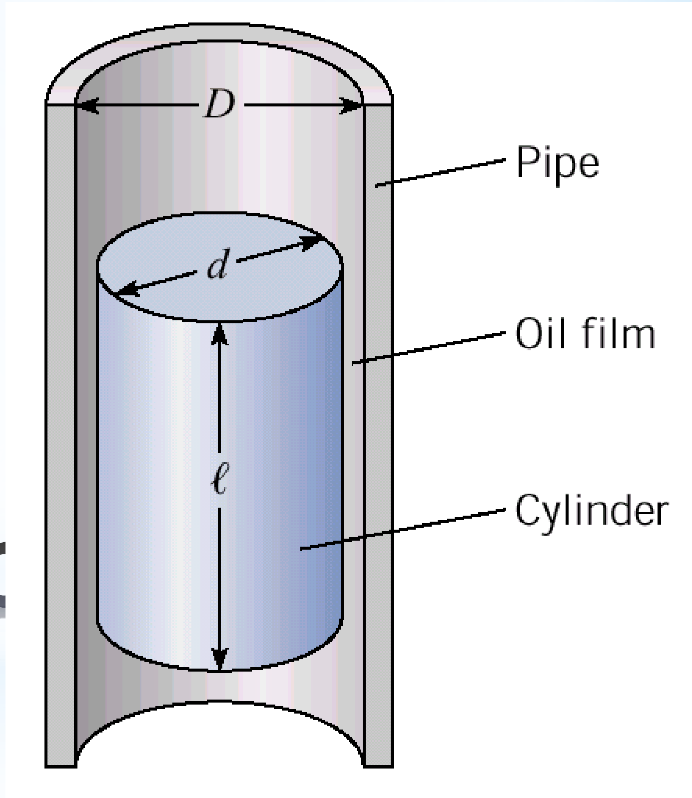
Flow between 2 plates

* 2 different coordinate systems

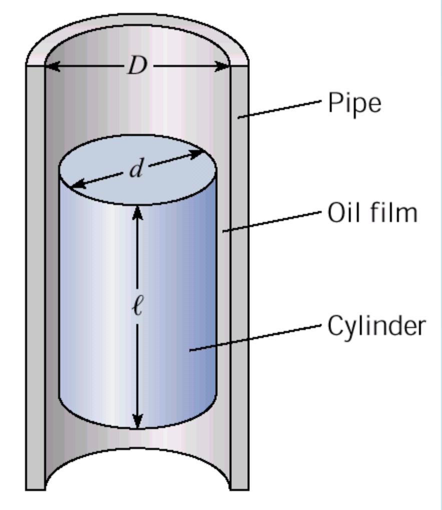


* Given
  * Rotation rate, ω = 1500 rpm
  * d = 6 cm
  * l = 40 cm
  * D = 6.02 cm
  * SG_{oil} = 0.88
  * ν_{oil} = 0.003 m^2/s
* Find: Torque and Power required to turn the bearing at the indicated speed.

Examp



* Example: cont.



* Assume: Linear velocity profile in oil film

$$\text{Shear Stress } \tau = \mu \frac{dV}{dy} = \mu \frac{\omega(d/2)}{(D-d)/2}$$

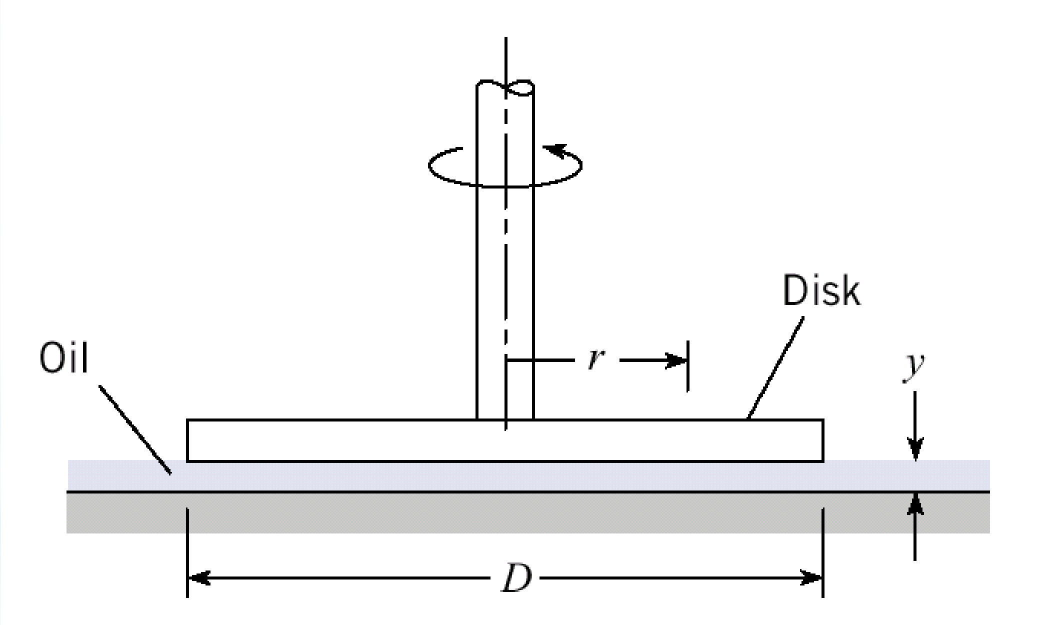
$$= (0.88 * 998 * 0.003) \frac{\left(\frac{2\pi}{60} * 1500\right)(0.06/2)}{(0.0002)/2} = 124\, kN/m^2$$

$$\text{Torque } M = (2\pi\tau \frac{d}{2} l)\frac{d}{2}$$

$$= (2\pi * 124{,}000 * \frac{0.06}{2} * 0.4)\frac{0.06}{2} = 281\, N \cdot m$$

$$\text{Power } P = M\omega = 281 * 157.1 = 44{,}100\, N \cdot m/s = 44.1\, kW$$

* Assume linear velocity profile: $dV/dy = V/y = \omega r/y$
* Find shear stress



ting Disk

Kinematics of Flow

A. KINEMATICS OF FLOW

▶ 5.1 INTRODUCTION

Kinematics is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in a flow field at any time is studied in this branch of fluid mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.

▶ 5.2 METHODS OF DESCRIBING FLUID MOTION

*Kinematics of Flow

▶ 5.3 TYPES OF FLUID FLOW

The fluid flow is classified as :

(*i*) Steady and unsteady flows ;
(*ii*) Uniform and non-uniform flows ;
(*iii*) Laminar and turbulent flows ;
(*iv*) Compressible and incompressible flows ;
(*v*) Rotational and irrotational flows ; and
(*vi*) One, two and three-dimensional flows.

*Kinematics of Flow

5.3.1 Steady and Unsteady Flows. Steady flow is defined as that type of flow in which the fluid characteristics like velocity, pressure, density, etc., at a point do not change with time. Thus for steady flow, mathematically, we have

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} = 0, \left(\frac{\partial \rho}{\partial t}\right)_{x_0, y_0, z_0} = 0$$

where (x_0, y_0, z_0) is a fixed point in fluid field.

Unsteady flow is that type of flow, in which the velocity, pressure or density at a point changes with respect to time. Thus, mathematically, for unsteady flow

$$\left(\frac{\partial V}{\partial t}\right)_{x_0, y_0, z_0} \neq 0, \left(\frac{\partial p}{\partial t}\right)_{x_0, y_0, z_0} \neq 0 \text{ etc.}$$

*Kinematics of Flow

5.3.2 Uniform and Non-uniform Flows. Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to space (*i.e.*, length of direction of the flow). Mathematically, for uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} = 0$$

where ∂V = Change of velocity

∂s = Length of flow in the direction S.

Non-uniform flow is that type of flow in which the velocity at any given time changes with respect to space. Thus, mathematically, for non-uniform flow

$$\left(\frac{\partial V}{\partial s}\right)_{t\,=\,\text{constant}} \neq 0.$$

*Kinematics of Flow

5.3.3 Laminar and Turbulent Flows. Laminar flow is defined as that type of flow in which the fluid particles move along well-defined paths or stream line and all the stream-lines are straight and parallel. Thus the particles move in laminas or layers gliding smoothly over the adjacent layer. This type of flow is also called stream-line flow or viscous flow.

Turbulent flow is that type of flow in which the fluid particles move in a *zig-zag* way. Due to the movement of fluid particles in a *zig-zag* way, the eddies formation takes place which are responsible for high energy loss. For a pipe flow, the type of flow is determined by a non-dimensional number $\frac{VD}{\nu}$ called the Reynold number,

where D = Diameter of pipe

V = Mean velocity of flow in pipe

and ν = Kinematic viscosity of fluid.

If the Reynold number is less than 2000, the flow is called laminar. If the Reynold number is more than 4000, it is called turbulent flow. If the Reynold number lies between 2000 and 4000, the flow may be laminar or turbulent.

*Kinematics of Flow

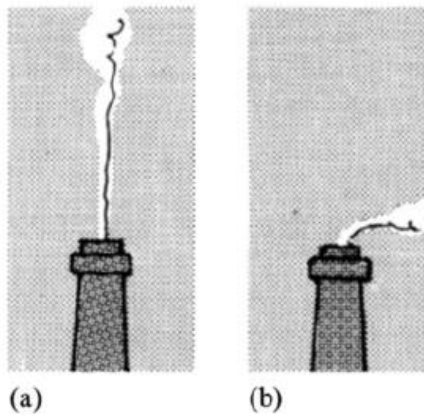


Fig. 4.4 Smoke from a chimney

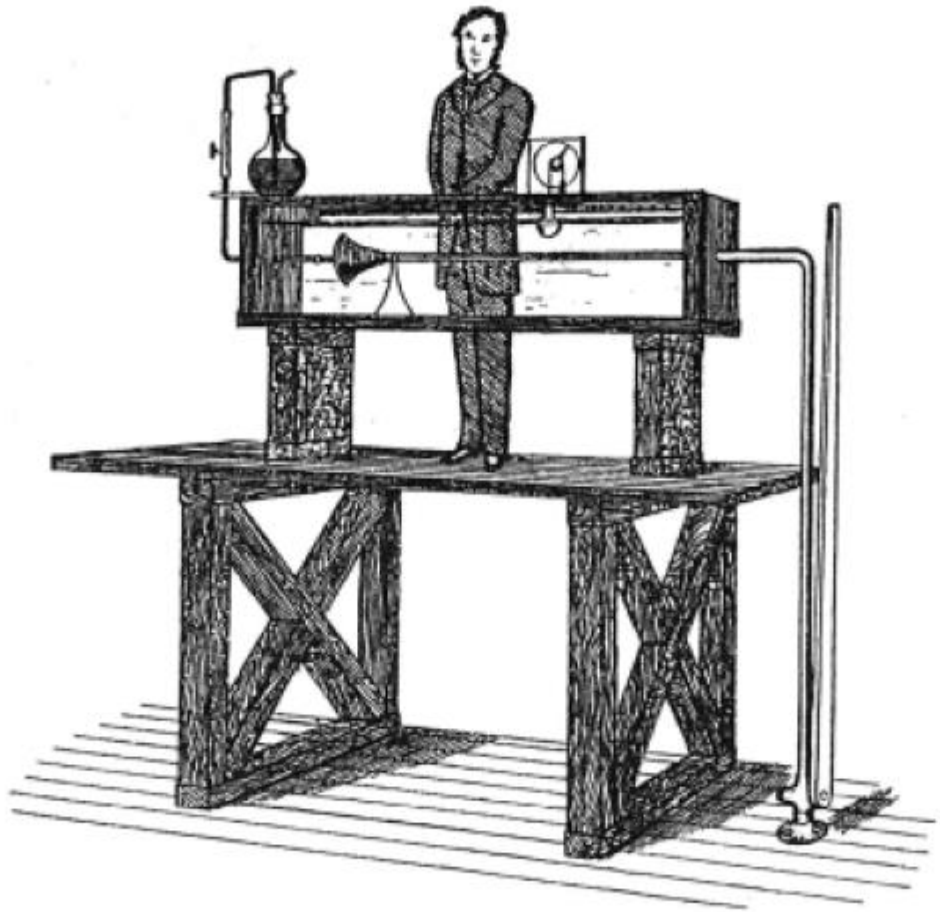


Fig. 4.5 Reynolds' experiment[1]

*Kinematics of Flow

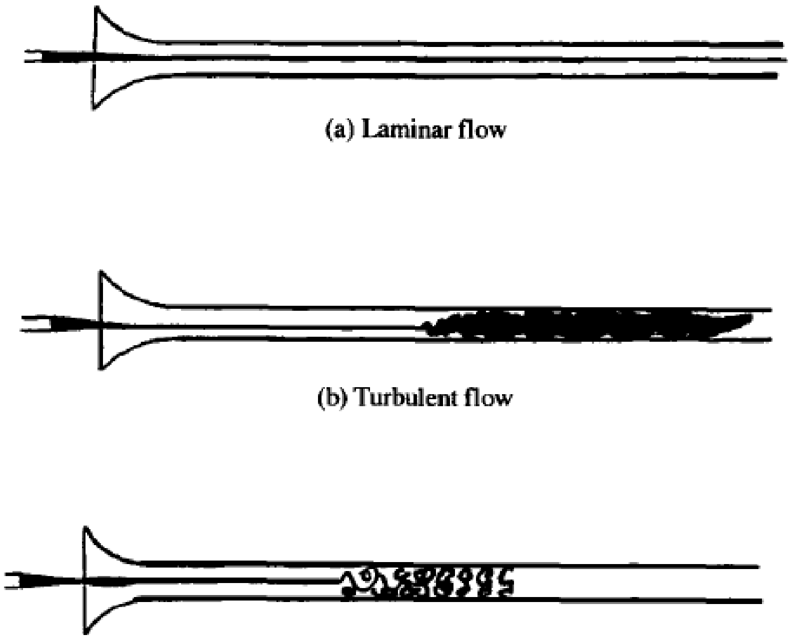


Fig. 4.6 Reynolds' sketch of transition from laminar flow to turbulent flow

Reynolds number

$$Re = \frac{\rho v d}{\mu} = \frac{vd}{\nu} \quad (\nu \text{ is the kinematic viscosity})$$

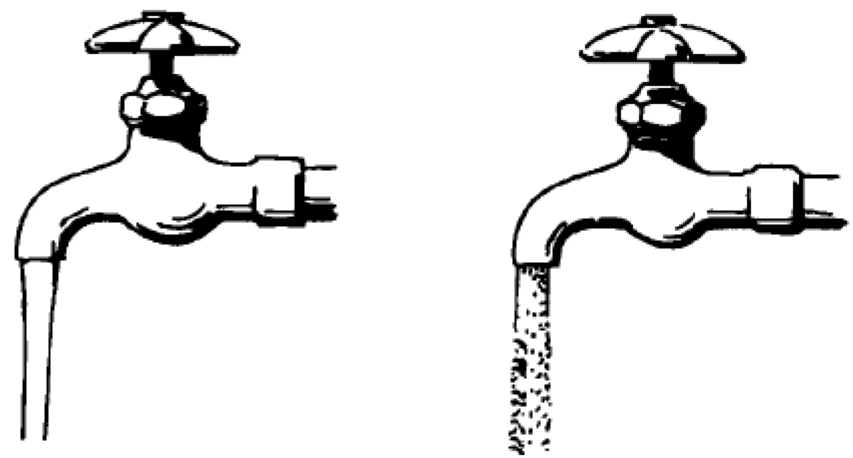


Fig. 4.7 Water flowing from a faucet

*Kinematics of Flow

5.3.4 Compressible and Incompressible Flows. Compressible flow is that type of flow in which the density of the fluid changes from point to point or in other words the density (ρ) is not constant for the fluid. Thus, mathematically, for compressible flow

$$\rho \neq \text{Constant}$$

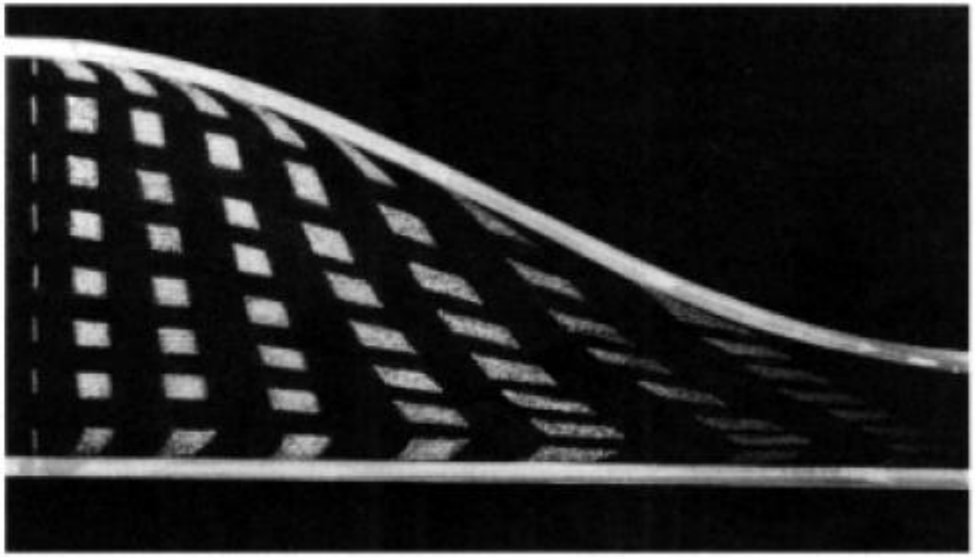
Incompressible flow is that type of flow in which the density is constant for the fluid flow. Liquids are generally incompressible while gases are compressible. Mathematically, for incompressible flow

$$\rho = \text{Constant.}$$

*Kinematics of Flow

5.3.5 Rotational and Irrotational Flows. Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis. And if the fluid particles while flowing along stream-lines, do not rotate about their own axis then that type of flow is called irrotational flow.

Fluid particles running through a narrow channel flow, while undergoing deformation and rotation, are shown in Fig. 4.8.



Deformation and rotation of fluid particles running through a narrowing channel

*Kinematics of Flow

5.3.6 One-, Two- and Three-Dimensional Flows. **One-dimensional flow** is that type of flow in which the flow parameter such as velocity is a function of time and one space co-ordinate only, say x. For a steady one-dimensional flow, the velocity is a function of one-space-co-ordinate only. The variation of velocities in other two mutually perpendicular directions is assumed negligible. Hence mathematically, for one-dimensional flow

$$u = f(x),\ v = 0 \text{ and } w = 0$$

where u, v and w are velocity components in x, y and z directions respectively.

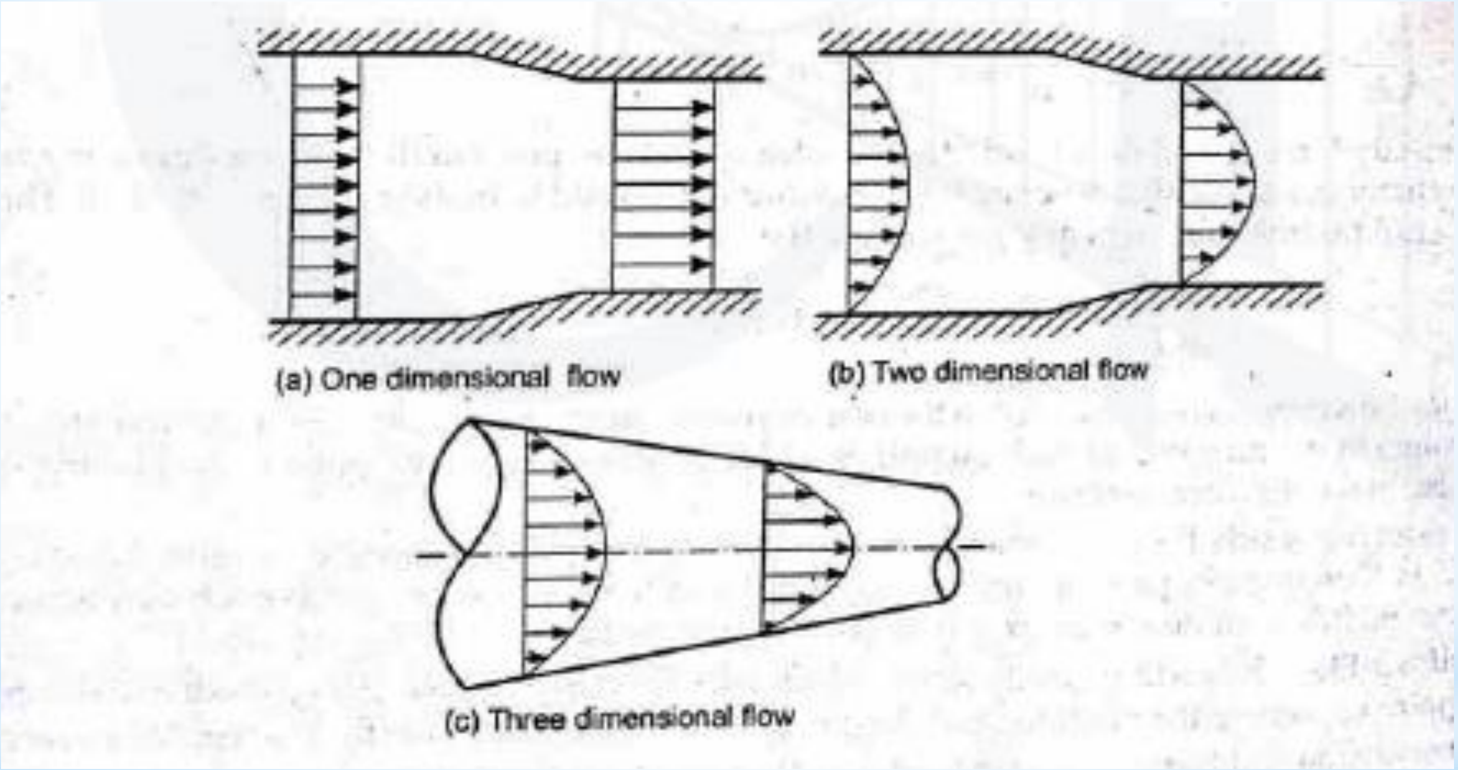
Two-dimensional flow is that type of flow in which the velocity is a function of time and two rectangular space co-ordinates say x and y. For a steady two-dimensional flow the velocity is a function of two space co-ordinates only. The variation of velocity in the third direction is negligible. Thus, mathematically for two-dimensional flow

$$u = f_1(x, y),\ v = f_2(x, y) \text{ and } w = 0.$$

Three-dimensional flow is that type of flow in which the velocity is a function of time and three mutually perpendicular directions. But for a steady three-dimensional flow the fluid parameters are functions of three space co-ordinates (x, y and z) only. Thus, mathematically, for three-dimensional flow

$$u = f_1(x, y, z),\ v = f_2(x, y, z) \text{ and } w = f_3(x, y, z).$$

*Kinematics of Flow



*Kinematics of Flow

▶ 5.4 RATE OF FLOW OR DISCHARGE (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or a channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus

(*i*) For liquids the units of Q are m^3/s or litres/s

(*ii*) For gases the units of Q is kgf/s or Newton/s

Consider a liquid flowing through a pipe in which

A = Cross-sectional area of pipe

V = Average velocity of fluid across the section

Then discharge $Q = A \times V$. ...(5.1)

*Kinematics of Flow

▶ 5.5 CONTINUITY EQUATION

The equation based on the principle of conservation of mass is called continuity equation. Thus for a fluid flowing through the pipe at all the cross-section, the quantity of fluid per second is constant. Consider two cross-sections of a pipe as shown in Fig. 5.1.

Let V_1 = Average velocity at cross-section 1-1

ρ_1 = Density at section 1-1

A_1 = Area of pipe at section 1-1

and V_2, ρ_2, A_2 are corresponding values at section, 2-2.

Then rate of flow at section 1-1 = $\rho_1 A_1 V_1$

Rate of flow at section 2-2 = $\rho_2 A_2 V_2$

According to law of conservation of mass

Rate of flow at section 1-1 = Rate of flow at section 2-2

or

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \qquad \text{...(5.2)}$$

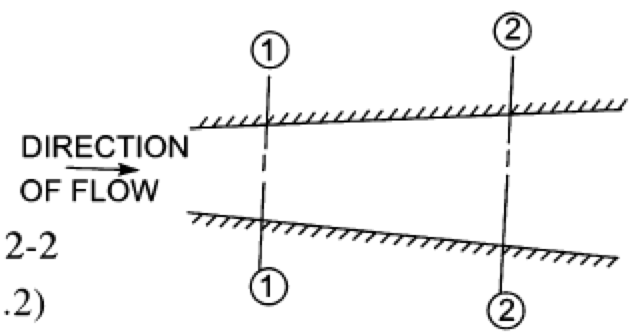


Fig. 5.1 *Fluid flowing through a pipe.*

Equation (5.2) is applicable to the compressible as well as incompressible fluids and is called **Continuity Equation**. If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation (5.2) reduces to

$$A_1 V_1 = A_2 V_2 \qquad \text{...(5.3)}$$

*Kinematics of Flow

▶ 5.6 CONTINUITY EQUATION IN THREE-DIMENSIONS

{$\because$ $dydz$ is constant}

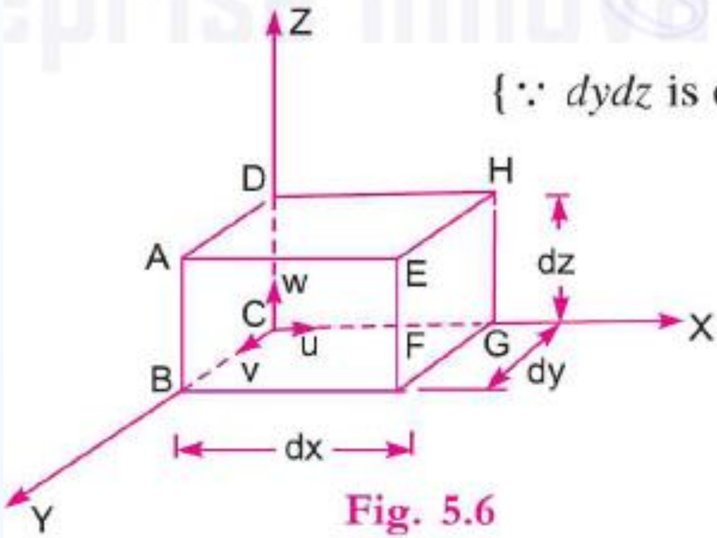


Fig. 5.6

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z directions respectively. Mass of fluid entering the face $ABCD$ per second

$$= \rho \times \text{Velocity in } x\text{-direction} \times \text{Area of } ABCD$$

$$= \rho \times u \times (dy \times dz)$$

Then mass of fluid leaving the face $EFGH$ per second $= \rho u\, dydz + \dfrac{\partial}{\partial x}(\rho u\, dydz)\, dx$

*Kinematics of Flow

$\therefore$ Gain of mass in x-direction

$$= \text{Mass through } ABCD - \text{Mass through } EFGH \text{ per second}$$

$$= \rho u\, dydz - \rho u\, dydz - \frac{\partial}{\partial x}(\rho u\, dydz)dx$$

$$= -\frac{\partial}{\partial x}(\rho u\, dydz)\, dx$$

$$= -\frac{\partial}{\partial x}(\rho u)\, dx\, dydz$$

Similarly, the net gain of mass in y-direction

$$= -\frac{\partial}{\partial y}(\rho v)\, dxdydz$$

and in z-direction $= -\frac{\partial}{\partial z}(\rho w)\, dxdydz$

$$\therefore \quad \text{Net gain of masses} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz$$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element. But mass of fluid in the element is $\rho.\, dx.\, dy.\, dz$ and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\ dx.\ dy.\ dz)$ or $\frac{\partial \rho}{\partial t}.\ dx\ dy\ dz.$

*Kinematics of Flow

Equating the two expressions,

or $$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right] dxdydz = \frac{\partial \rho}{\partial t}.\ dxdydz$$

or $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \text{ [Cancelling } dx.dy.dz \text{ from both sides]} \quad ...(5.3A)$$

Equation (5.3*A*) is the continuity equation in cartesian co-ordinates in its most general form. This equation is applicable to :

(*i*) Steady and unsteady flow,

(*ii*) Uniform and non-uniform flow, and

(*iii*) Compressible and incompressible fluids.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$ and hence equation (5.3*A*) becomes as

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w) = 0 \quad ...(5.3B)$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \quad ...(5.4)$$

Equation (5.4) is the continuity equation in three-dimensions. For a two-dimensional flow, the component $w = 0$ and hence continuity equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0. \quad ...(5.5)$$

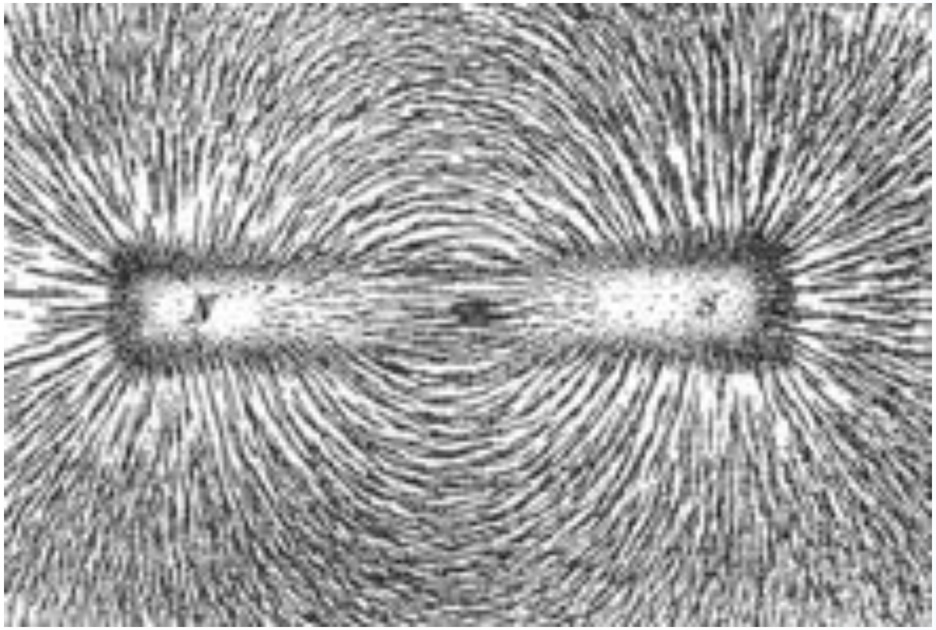
*Kinematics of Flow

Streamlines: are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction in which a massless fluid element will travel at any point in time.

Streaklines: are the loci of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline.

Pathlines: are the trajectories that individual fluid particles follow. These can be thought of as "recording" the path of a fluid element in the flow over a certain period. The direction the path takes will be determined by the streamlines of the fluid at each moment in time.

*Kinematics of Flow



The direction of magnetic field lines are streamlines represented by the alignment of iron filings sprinkled on paper placed above a bar magnet

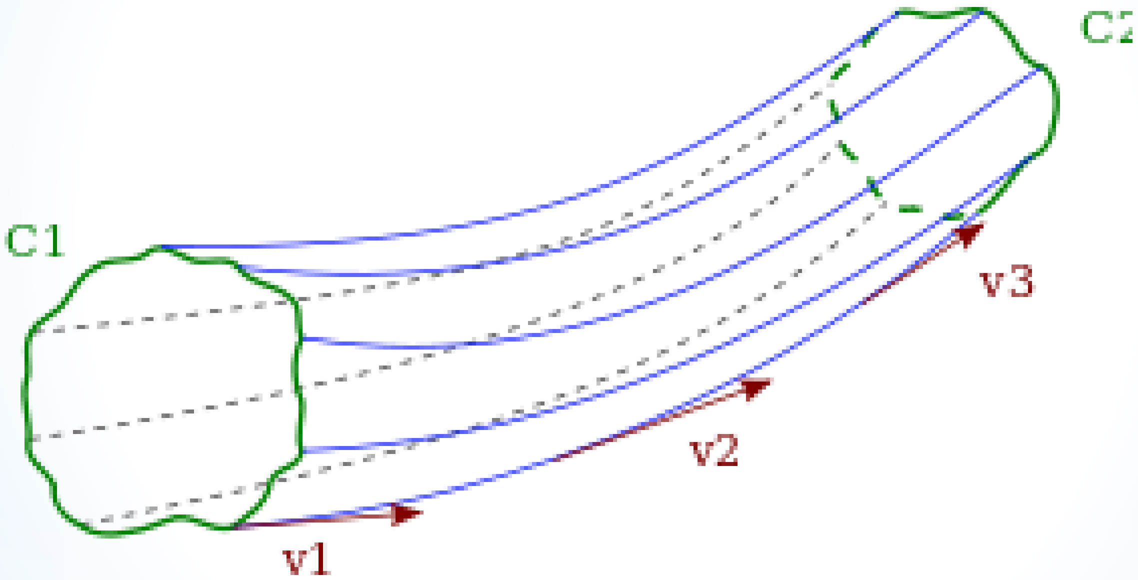
*Kinematics of Flow

A long-exposure photo of spark from a campfire shows the pathlines for the flow of hot air.

*Kinematics of Flow

Example of a streakline used to visualize the flow around a car inside a wind tunnel.

*Kinematics of Flow



Solid blue lines and broken grey lines represent the streamlines. The red arrows show the direction and magnitude of the flow velocity. These arrows are tangential to the streamline. The group of streamlines enclose the green curves to form a stream surface.

*Kinematics of Flow

▶ 5.9 TYPES OF MOTION

A fluid particle while moving may undergo anyone or combination of following four types of displacements :

(*i*) Linear Translation or Pure Translation,
(*ii*) Linear Deformation,
(*iii*) Angular Deformation, and
(*iv*) Rotation.

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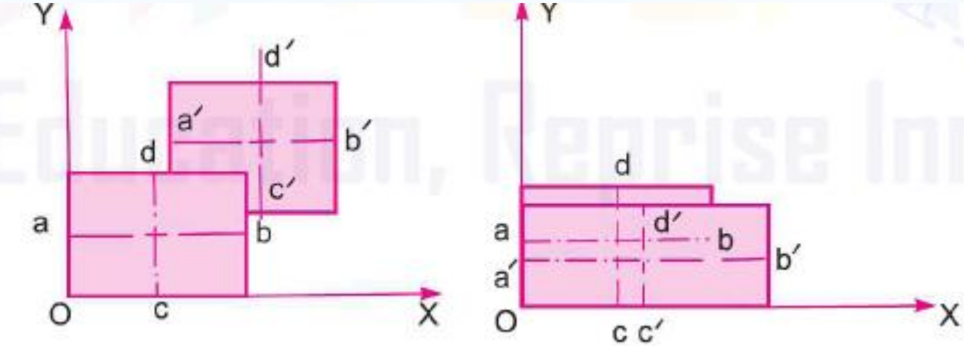

(a) LINEAR TRANSLATION (b) LINEAR DEFORMATION

5.9.1 Linear Translation. It is defined as the movement of a fluid element in such a way that it moves bodily from one position to another position and the two axes *ab* and *cd* represented in new positions by *a′b′* and *c′d′* are parallel as shown in Fig. 5.11 (*a*).

5.9.2 Linear Deformation. It is defined as the deformation of a fluid element in linear direction when the element moves. The axes of the element in the deformed position and un-deformed position are parallel, but their lengths change as shown in Fig. 5.11 (*b*).

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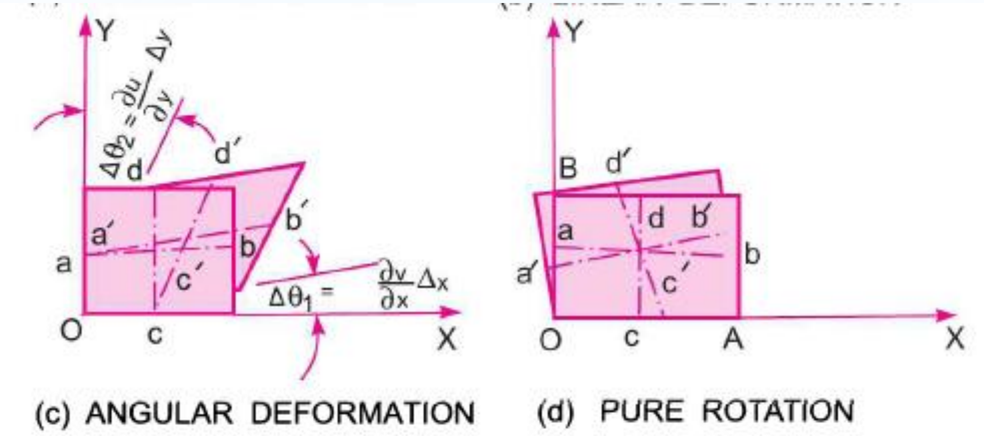


Fig. 5.11. *Displacement of a fluid element.*

*Kinematics of Flow

5.9.3 Angular Deformation or Shear Deformation. It is defined as the average change in the angle contained by two adjacent sides. Let $\Delta\theta_1$ and $\Delta\theta_2$ is the change in angle between two adjacent sides of a fluid element as shown in Fig. 5.11 (*c*), then angular deformation or shear strain rate

$$= \frac{1}{2}\,[\Delta\theta_1 + \Delta\theta_2]$$

Now $$\Delta\theta_1 = \frac{\partial v}{\partial x} \times \frac{\Delta x}{\Delta x} = \frac{\partial v}{\partial x} \text{ and } \Delta\theta_2 = \frac{\partial u}{\partial y} \cdot \frac{\Delta y}{\Delta y} = \frac{\partial u}{\partial y}.$$

∴ $$\textbf{Angular deformation} = \frac{1}{2}\,[\Delta\theta_1 + \Delta\theta_2]$$

or $$\textbf{Shear strain rate} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] \quad \text{...(5.16)}$$

*Kinematics of Flow

5.9.4 Rotation. It is defined as the movement of a fluid element in such a way that both of its axes (horizontal as well as vertical) rotate in the same direction as shown in Fig. 5.11 (*d*). It is equal to $\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ for a two-dimensional element in *x*-*y* plane. The rotational components are

$$\left.\begin{aligned}\omega_z &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\\ \omega_x &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\\ \omega_y &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\end{aligned}\right\} \quad \text{...(5.17)}$$

5.9.5 Vorticity. It is defined as the value twice of the rotation and hence it is given as 2ω.

*Kinematics of Flow

5.9.5 Vorticity. It is defined as the value twice of the rotation and hence it is given as 2ω.

Problem 5.18 *A fluid flow is given by $V = 8x^3 i - 10x^2 y j$.*

Find the shear strain rate and state whether the flow is rotational or irrotational.

Solution. Given : $V = 8x^3 i - 10x^2 y j$

$$\therefore \quad u = 8x^3, \quad \frac{\partial u}{\partial x} = 24x^2, \quad \frac{\partial u}{\partial y} = 0$$

and

$$v = -10x^2 y, \quad \frac{\partial v}{\partial x} = -20xy, \quad \frac{\partial v}{\partial y} = -10x^2$$

(*i*) Shear strain rate is given by equation (5.16) as

$$= \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}(-20xy + 0) = \mathbf{-10xy}. \textbf{ Ans.}$$

(*ii*) Rotation in x - y plane is given by equation (5.17) or

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(-20xy - 0) = -10xy$$

As rotation $\omega_z \neq 0$. Hence flow is rotational. **Ans.**

*Kinematics of Flow

5.9. CIRCULATION AND VORTICITY

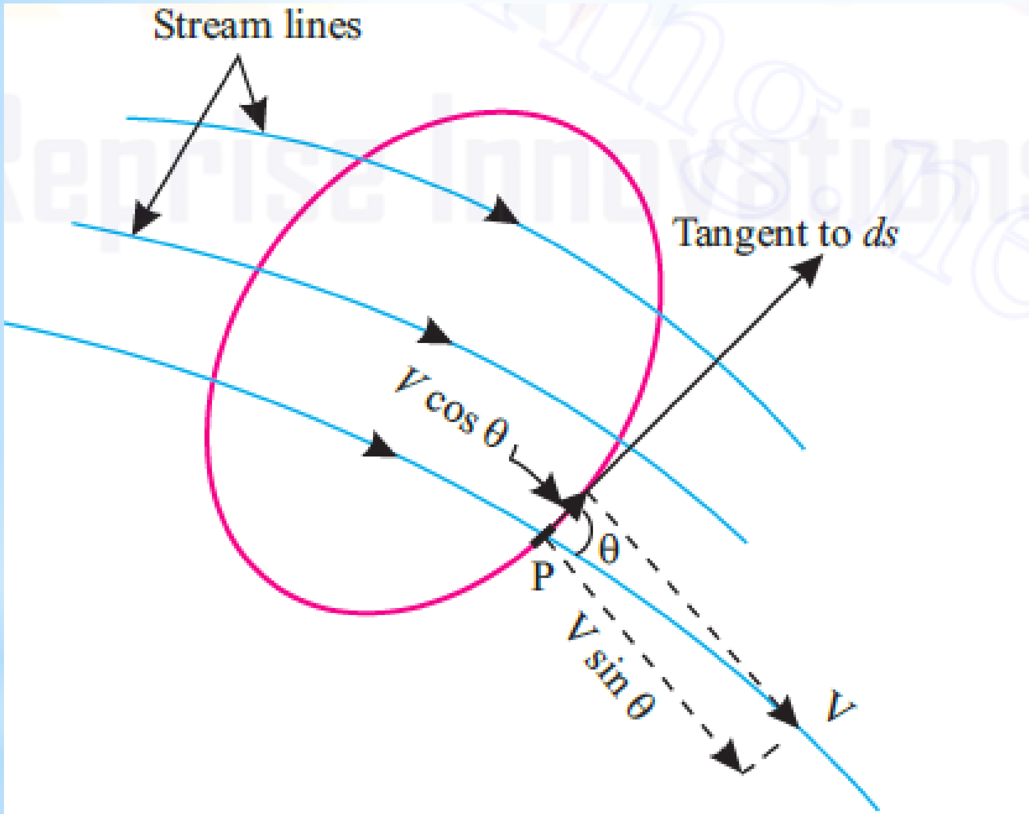


Fig. 5.18. Circulation in a two-dimensional flow.

Let us consider a closed curve in a two-dimensional flow field shown in Fig. 5.18; the curve being cut by the stream lines. Let P be the point of intersection of the curve with one stream line, θ be the angle which the stream line makes with the curve. The component of velocity along the closed curve at the point of intersection is equal to $V \cos \theta$. **Circulation** Γ *is defined mathematically as the line integral of the tangential velocity about a closed path* (*contour*).

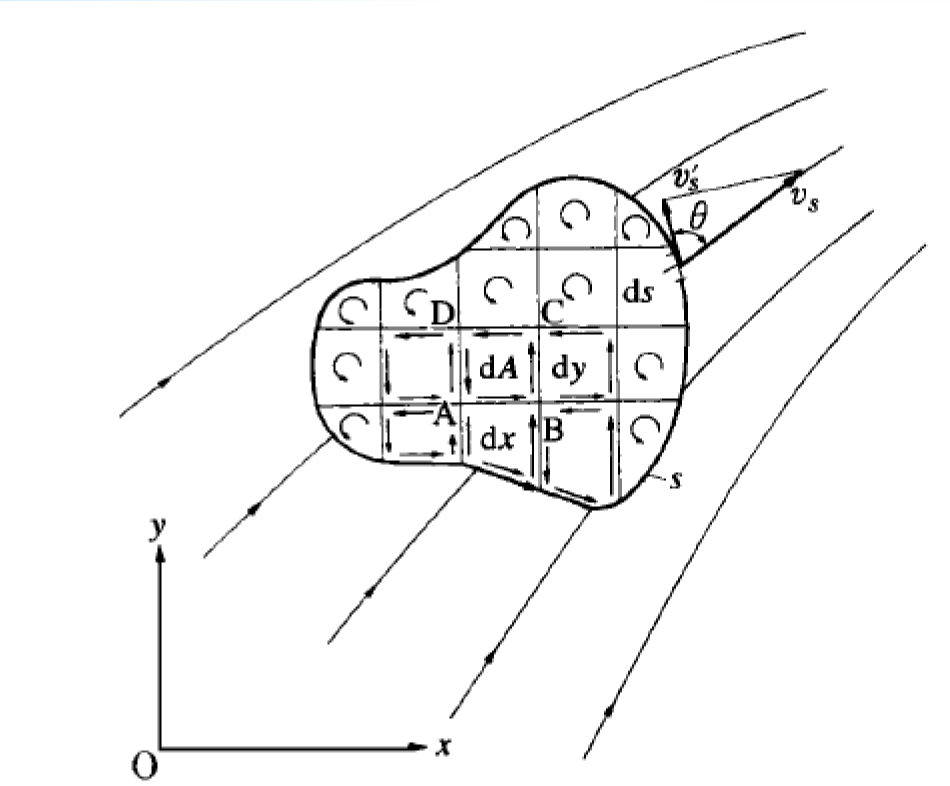
Thus,

$$\Gamma = \oint V \cos \theta . ds$$

where, V = Velocity in the flow field at the element ds, and

θ = Angle between V and tangent to the path (in the positive anticlockwise direction along the path) at that point.

*Kinematics of Flow



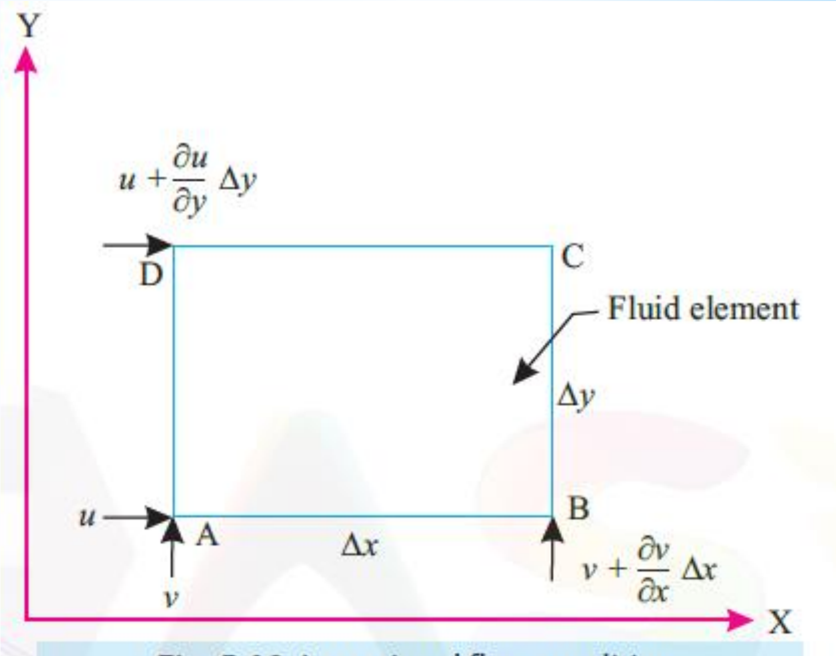


Fig. 5.19. Irrotational flow condition.

Circulation around regular curves can be obtained by integration. Let us consider the circulation around an elementary box (fluid element ABCD) shown in Fig. 5.19.

Starting from *A* and proceeding anticlockwise, we have:

$$d\,\Gamma = u\,\Delta x + \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta y - \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta x - v\,\Delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x.\Delta y$$

*Kinematics of Flow

The **vorticity** (Ω) is defined as the *circulation per unit of enclosed area,*

$$\Omega = \frac{\Gamma}{A}. \text{ Thus},$$

$$\Omega = \frac{d\Gamma}{\Delta x . \Delta y} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad ...(5.29)$$

If *a flow possesses vorticity, it is rotational.* **Rotation** ω (*omega*) *is defined as one-half of the vorticity*, or

$$\omega = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$$

The flow is irrotational if rotation ω *is zero.*

*Kinematics of Flow

5.8.2 Stream Function. It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (*Psi*) and defined only for two-dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = v$$

and

$$\frac{\partial \psi}{\partial y} = -u$$

The continuity equation for two-dimensional flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$

Substituting the values of u and v from equation (5.12), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0 \text{ or } -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} = 0.$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Substituting the values of u and v from equation (5.12) in the above rotational component, we get

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right] = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right]$$

*Kinematics of Flow

For irrotational flow, $\omega_z = 0$. Hence above equation becomes as $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial y^2} = 0$

which is Laplace equation for ψ.

The **properties** of stream function (ψ) are :

1. If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational.
2. If stream function (ψ) satisfies the Laplace equation, it is a possible case of an irrotational flow.

*Kinematics of Flow

5.8.1 Velocity Potential Function. It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ (Phi). Mathematically, the velocity, potential is defined as $\phi = f(x, y, z)$ for steady flow such that

$$\left.\begin{aligned} u &= -\frac{\partial \phi}{\partial x} \\ v &= -\frac{\partial \phi}{\partial y} \\ w &= -\frac{\partial \phi}{\partial z} \end{aligned}\right\}$$

where u, v and w are the components of velocity in x, y and z directions respectively.

The continuity equation for an incompressible steady flow is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$.

Substituting the values of u, v and w from equation (5.9), we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Equation (5.10) is a Laplace equation.

*Kinematics of Flow

For two-dimension case, equation (5.10) reduces to $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$ ...(5.11)

If any value of ϕ that satisfies the Laplace equation, will correspond to some case of fluid flow.

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x}\right]$$

$$\omega_y = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z \partial x} + \frac{\partial^2 \phi}{\partial x \partial z}\right]$$

$$\omega_x = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right] = \frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y \partial z} + \frac{\partial^2 \phi}{\partial z \partial y}\right]$$

If ϕ is a continuous function, then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$; $\frac{\partial^2 \phi}{\partial z \partial x} = \frac{\partial^2 \phi}{\partial x \partial z}$; etc.

$\therefore$ $\omega_z = \omega_y = \omega_x = 0.$

*Kinematics of Flow

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are :

1. If velocity potential (ϕ) exists, the flow should be irrotational.

2. If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

*Kinematics of Flow

5.8.5 Flow Net. A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. The flow net is an important tool in analysing two-dimensional irrotational flow problems.

Fig 5.22. shows some typical flow nets.

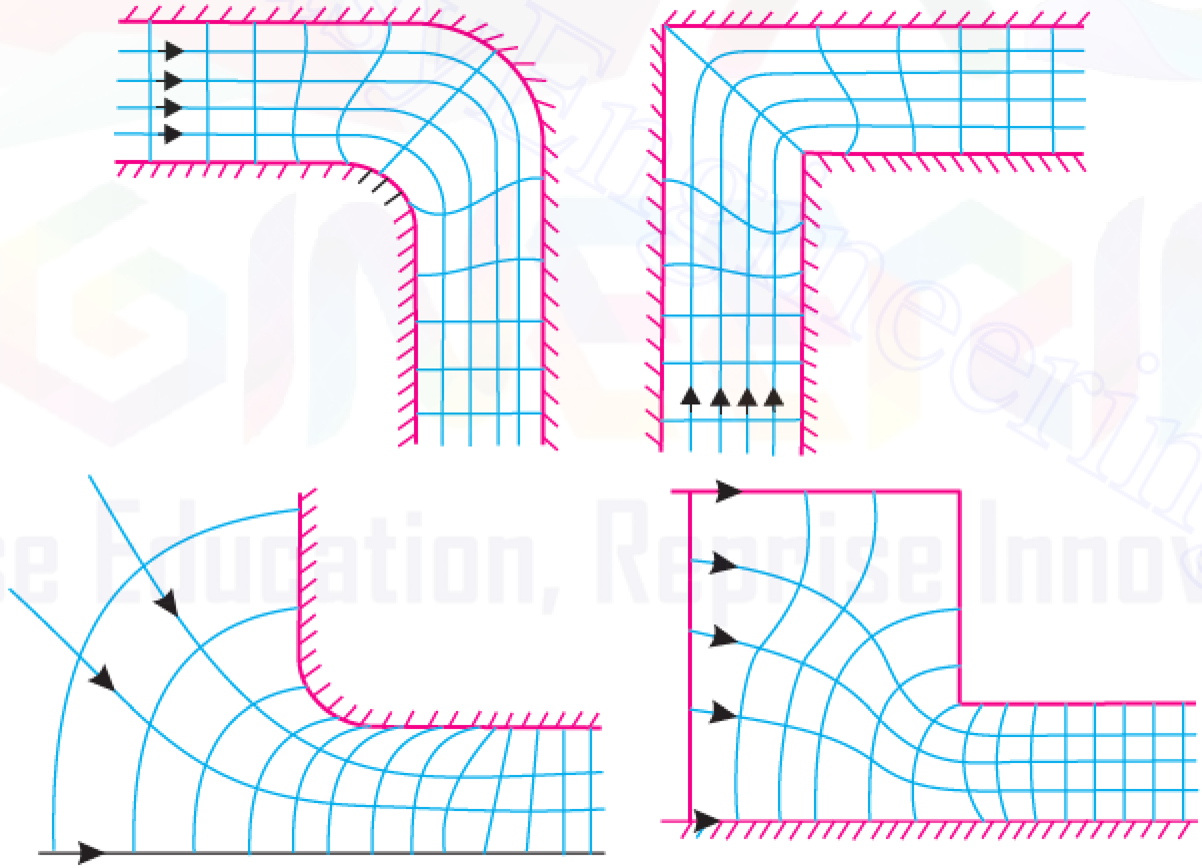


Fig. 5.22. Typical flow nets.

*Kinematics of Flow

▶ 5.10 VORTEX FLOW

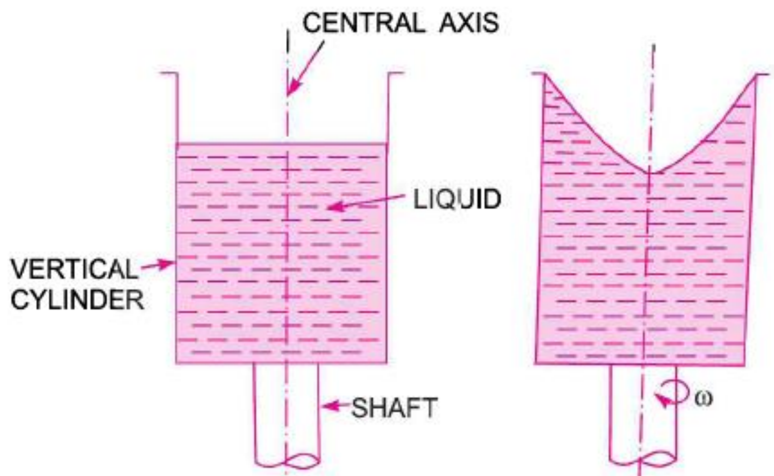
Vortex flow is defined as the flow of a fluid along a curved path or the flow of a rotating mass of fluid is known a 'Vortex Flow'. The vortex flow is of two types namely :

1. Forced vortex flow, and
2. Free vortex flow.

5.10.1 Forced Vortex Flow. Forced vortex flow is defined as that type of vortex flow, in which some external torque is required to rotate the fluid mass. The fluid mass in this type of flow, rotates at constant angular velocity, ω. The tangential velocity of any fluid particle is given by

$$v = \omega \times r \qquad \text{...(5.18)}$$

where r = Radius of fluid particle from the axis of rotation.



(a) CYLINDER IS STATIONARY (b) CYLINDER IS ROTATING

*Kinematics of Flow

Hence angular velocity ω is given by

$$\omega = \frac{v}{r} = \text{Constant.}$$

Examples of forced vortex are :

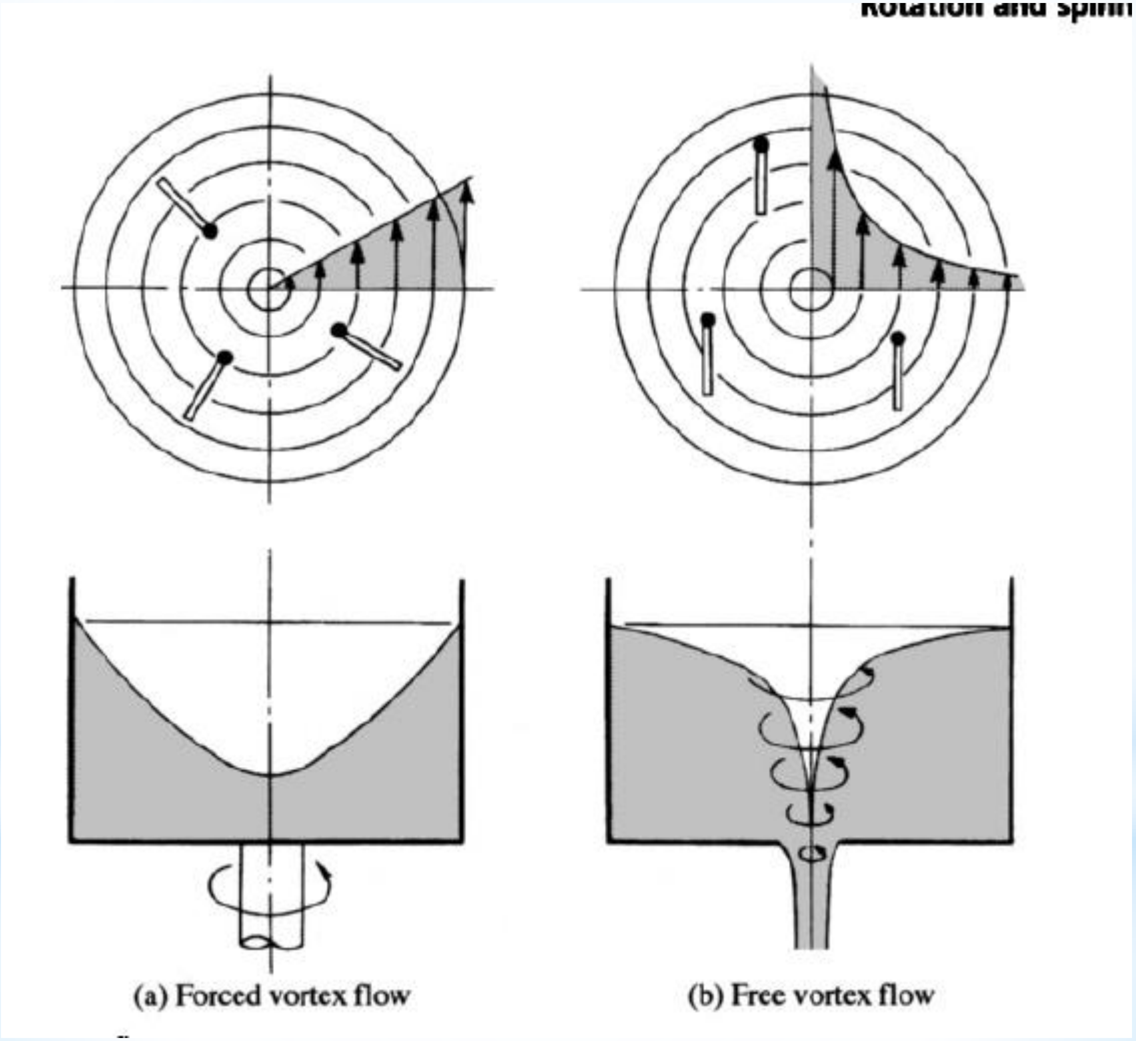
1. A vertical cylinder containing liquid which is rotated about its central axis with a constant angular velocity ω, as shown in Fig. 5.12.
2. Flow of liquid inside the impeller of a centrifugal pump.
3. Flow of water through the runner of a turbine.

5.10.2 Free Vortex Flow. When no external torque is required to rotate the fluid mass, that type of flow is called free vortex flow. Thus the liquid in case of free vortex is rotating due to the rotation which is imparted to the fluid previously.

Examples of the free vortex flow are :

1. Flow of liquid through a hole provided at the bottom of a container.
2. Flow of liquid around a circular bend in a pipe.
3. A whirlpool in a river.
4. Flow of fluid in a centrifugal pump casing.

*Kinematics of Flow



*Kinematics of Flow

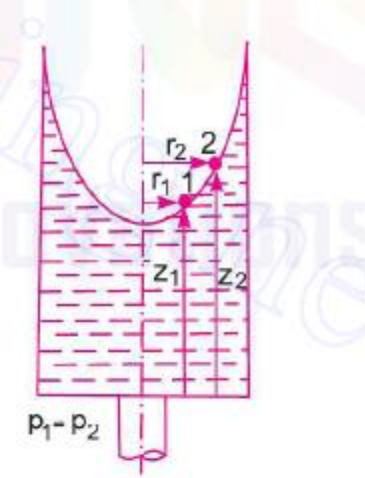
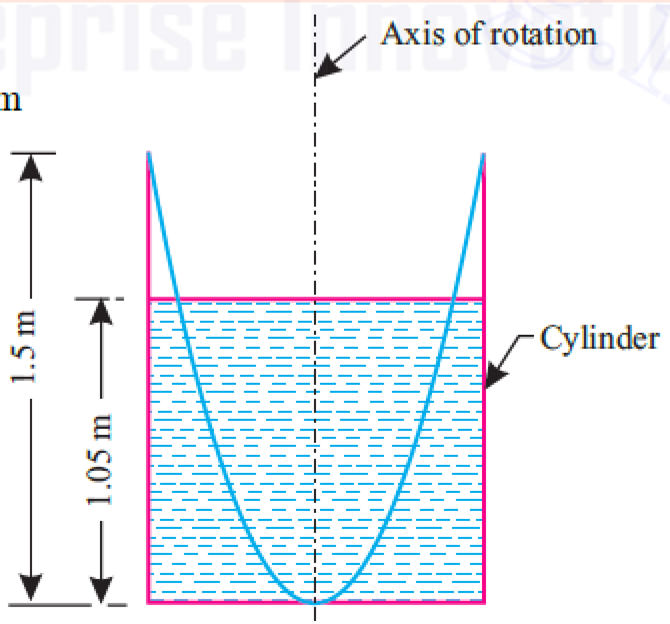


Fig. 5.15

$$[z_2 - z_1] = \frac{1}{2g}\,[v_2^{\,2} - v_1^{\,2}].$$

If the point 1 lies on the axis of rotation,

$$\therefore Z = \frac{v_2^{\,2}}{2g} = \frac{\omega^2 \times r_2^{\,2}}{2g}$$



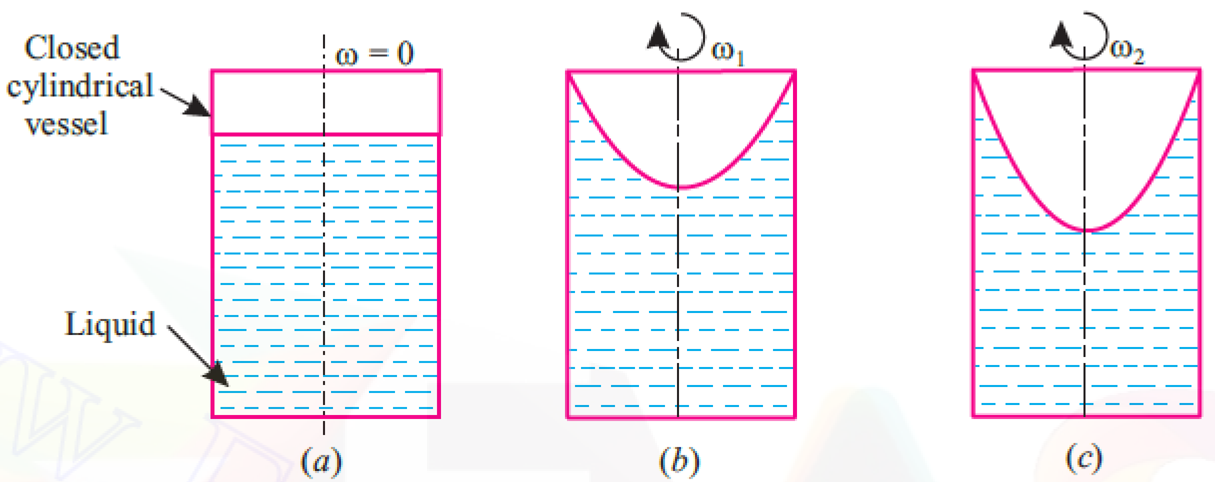
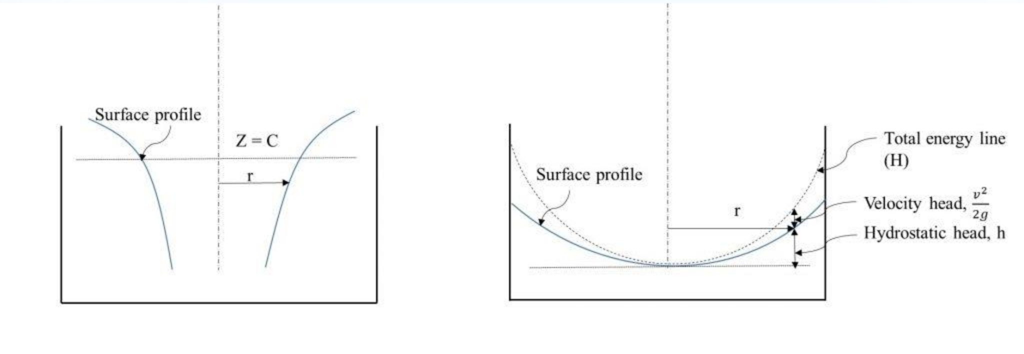


Fig. 6.72. Rotation of liquid in a closed cylindrical vessel.

*Kinematics of Flow



BUOYANCY AND FLOATATION

▶ 4.2 BUOYANCY

When a body is immersed in a fluid, an upward force is exerted by the fluid on the body. This upward force is equal to the weight of the fluid displaced by the body and is called the force of buoyancy or simply buoyancy.

by *Archimedes' principle* which states as follows:

"When a body is immersed in a fluid either wholly or partially, it is buoyed or lifted up by a force, which is equal to the weight of fluid displaced by the body."

▶ 4.3 CENTRE OF BUOYANCY

It is defined as the point, through which the force of buoyancy is supposed to act. As the force of buoyancy is a vertical force and is equal to the weight of the fluid displaced by the body, the centre of buoyancy will be the centre of gravity of the fluid displaced.

BUOYANCY AND FLOATATION

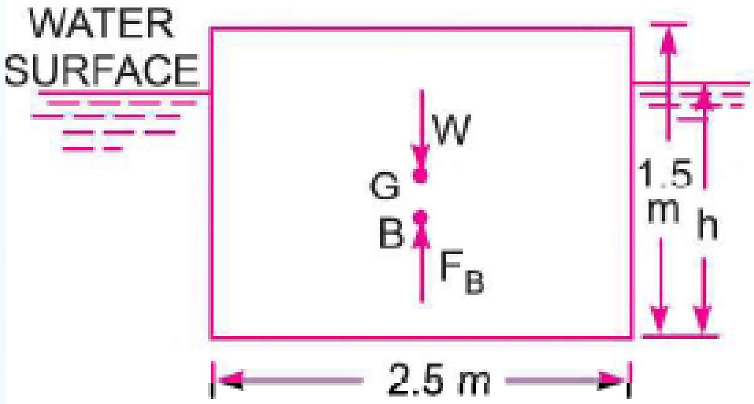


Fig. 4.1

BUOYANCY AND FLOATATION

4.7.1 Stability of a Sub-merged Body.

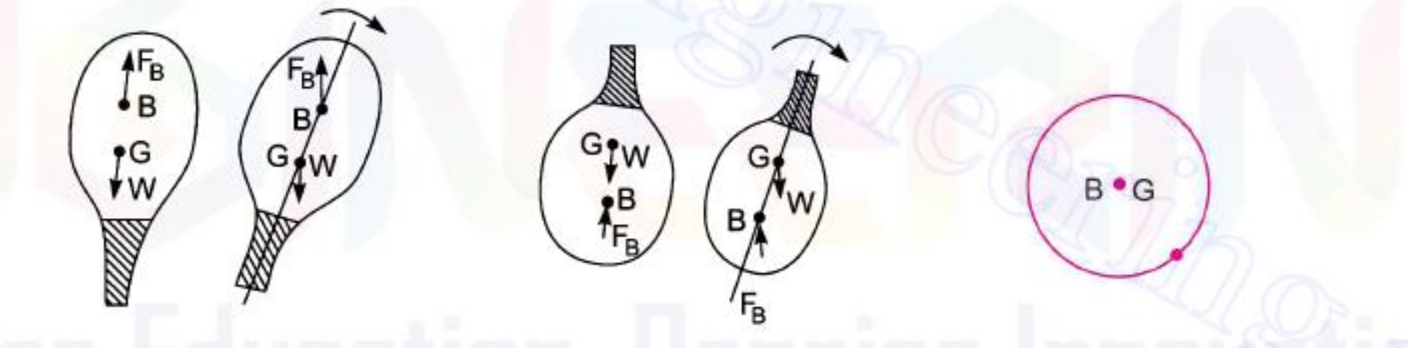


Fig. 4.12 *Stabilities of sub-merged bodies.*

(*a*) **Stable Equilibrium.** When $W = F_B$ and point B is above G, the body is said to be in stable equilibrium.

(*b*) **Unstable Equilibrium.** If $W = F_B$, but the centre of buoyancy (B) is below centre of gravity (G), the body is in unstable equilibrium as shown in Fig. 4.12 (*b*). A slight displacement to the body, in the clockwise direction, gives the couple due to W and F_B also in the clockwise direction. Thus the body does not return to its original position and hence the body is in unstable equilibrium.

(*c*) **Neutral Equilibrium.** If $F_B = W$ and B and G are at the same point, as shown in Fig. 4.12 (*c*), the body is said to be in neutral equilibrium.

BUOYANCY AND FLOATATION

▶ 4.4 META-CENTRE

It is defined as the point about which a body starts oscillating when the body is tilted by a small angle. The meta-centre may also be defined as the point at which the line of action of the force of buoyancy will meet the normal axis of the body when the body is given a small angular displacement.

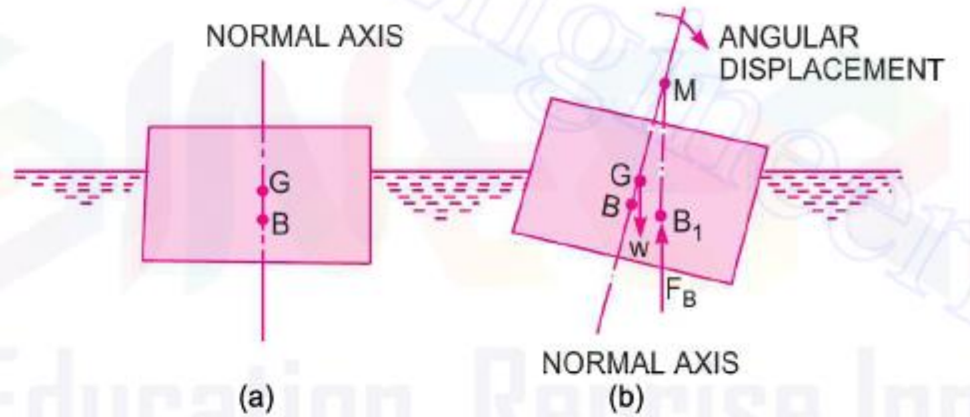


Fig. 4.5 *Meta-centre*

This point M is called **Meta-centre.**

BUOYANCY AND FLOATATION

▶ 4.5 META-CENTRIC HEIGHT

The distance *MG*, *i.e.*, the distance between the meta-centre of a floating body and the centre of gravity of the body is called meta-centric height.

BUOYANCY AND FLOATATION

4.2. TYPES OF EQUILIBRIUM OF FLOATING BODIES

The equilibrium of floating bodies is of the following types:

1. Stable equilibrium,
2. Unstable equilibrium, and
3. Neutral equilibrium.

4.3.1. Stable Equilibrium

When a body is given a small angular displacement (i.e. tilted slightly), by some external force, and then it returns back to its original position due to the internal forces (the weight and the upthrust), such an equilibrium is called **stable equilibrium**.

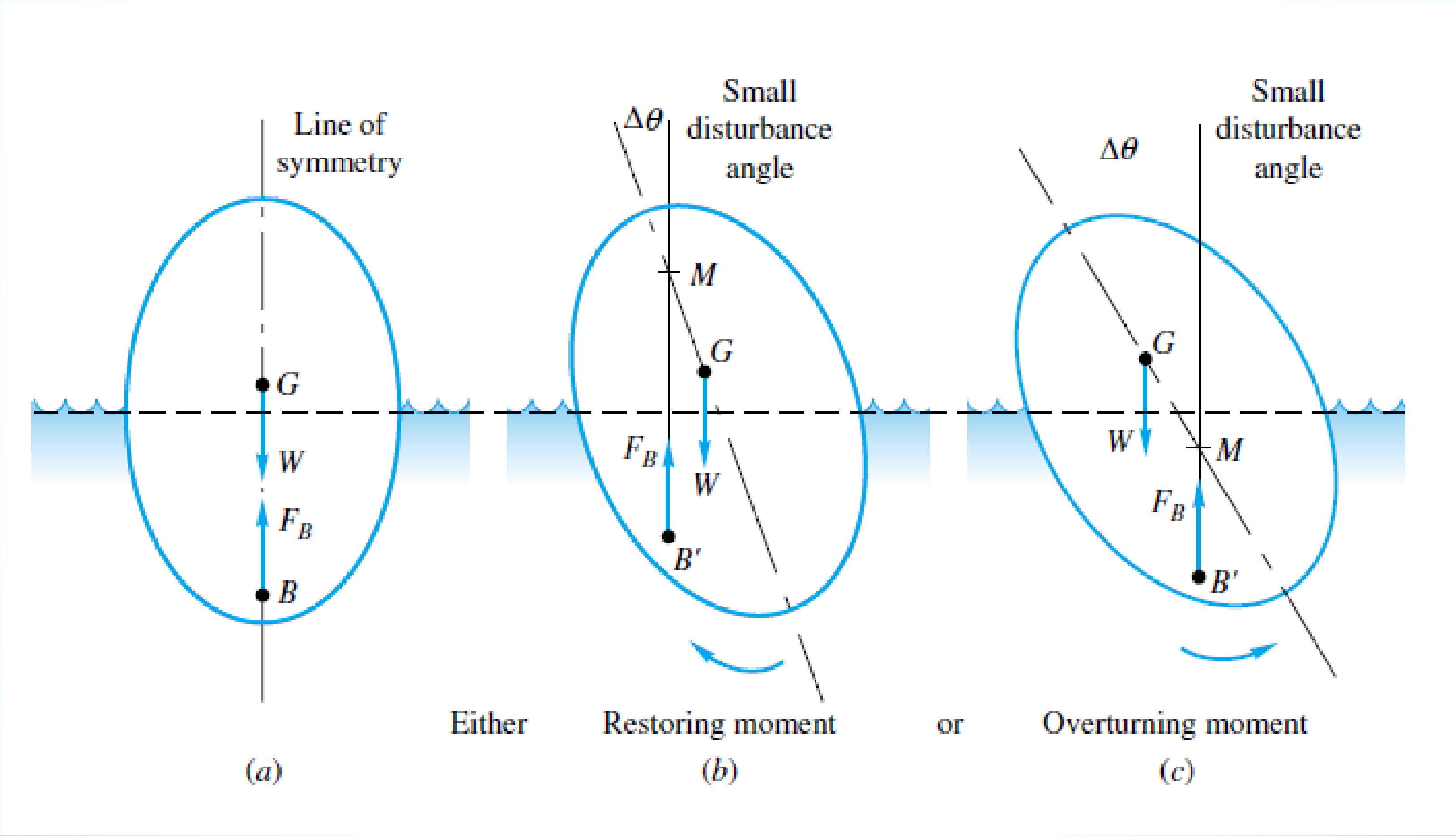
4.3.2. Unstable Equilibrium

If the body does not return to its original position from the slightly displaced angular position and heels farther away, when given a small angular displacement, such an equilibrium is called an **unstable equilibrium.**

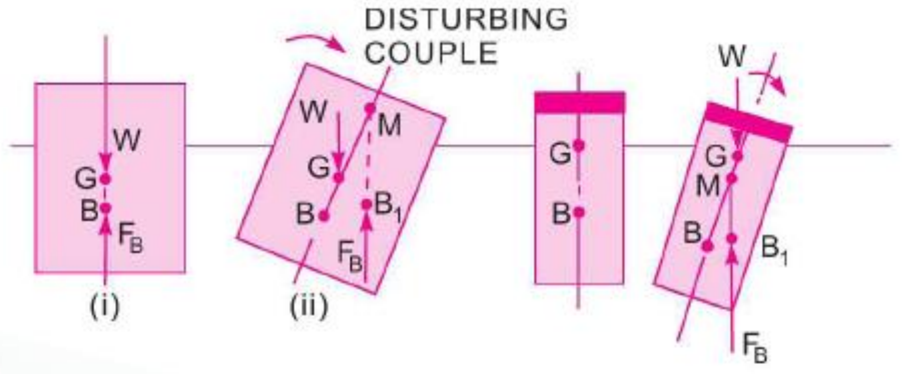
4.3.3. Neutral Equilibrium

If a body, when given a small angular displacement, occupies a new position and remains at rest in this new position, it is said to possess a **neutral equilibrium.**

BUOYANCY AND FLOATATION



BUOYANCY AND FLOATATION



(a) Stable equilibrium *M* is above *G* (b) Unstable equilibrium *M* is below *G*.

Fig. 4.13 *Stability of floating bodies.*

BUOYANCY AND FLOATATION

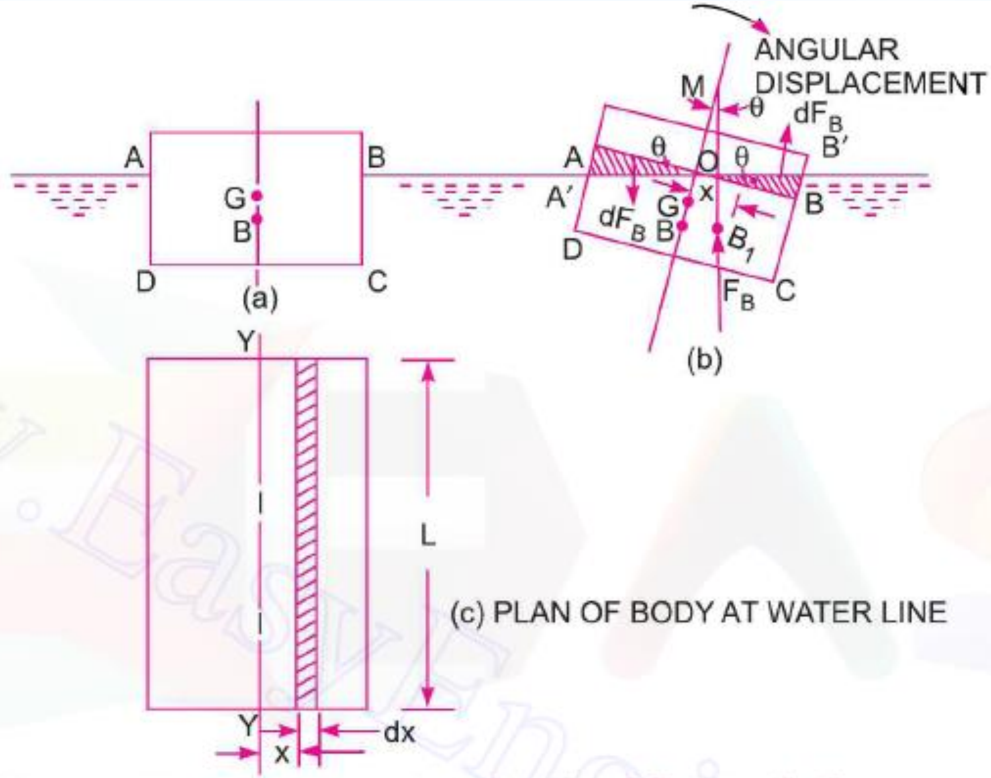


Fig. 4.6 *Meta-centre height of floating body.*

$$\therefore \quad \text{Meta-centric height} \quad = GM = \frac{I}{\forall} - BG.$$

BUOYANCY AND FLOATATION

▶ 4.8 EXPERIMENTAL METHOD OF DETERMINATION OF META-CENTRIC HEIGHT

The meta-centric height of a floating vessel can be determined, provided we know the centre of gravity of the floating vessel. Let w_1 is a known weight placed over the centre of the vessel as shown in Fig. 4.23 (*a*) and the vessel is floating.

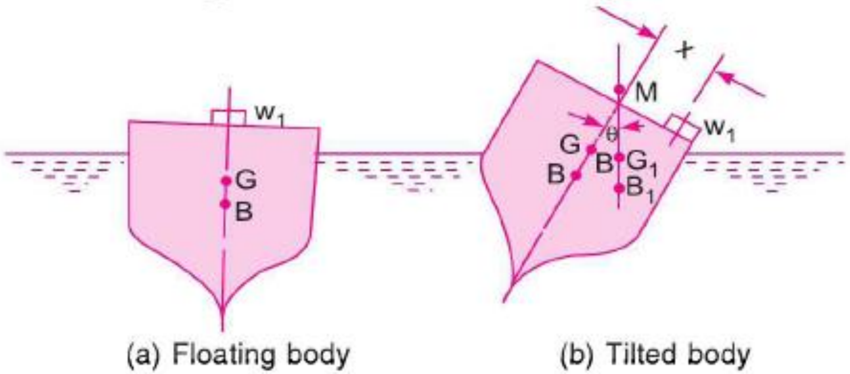


Fig. 4.23 *Meta-centric height.*

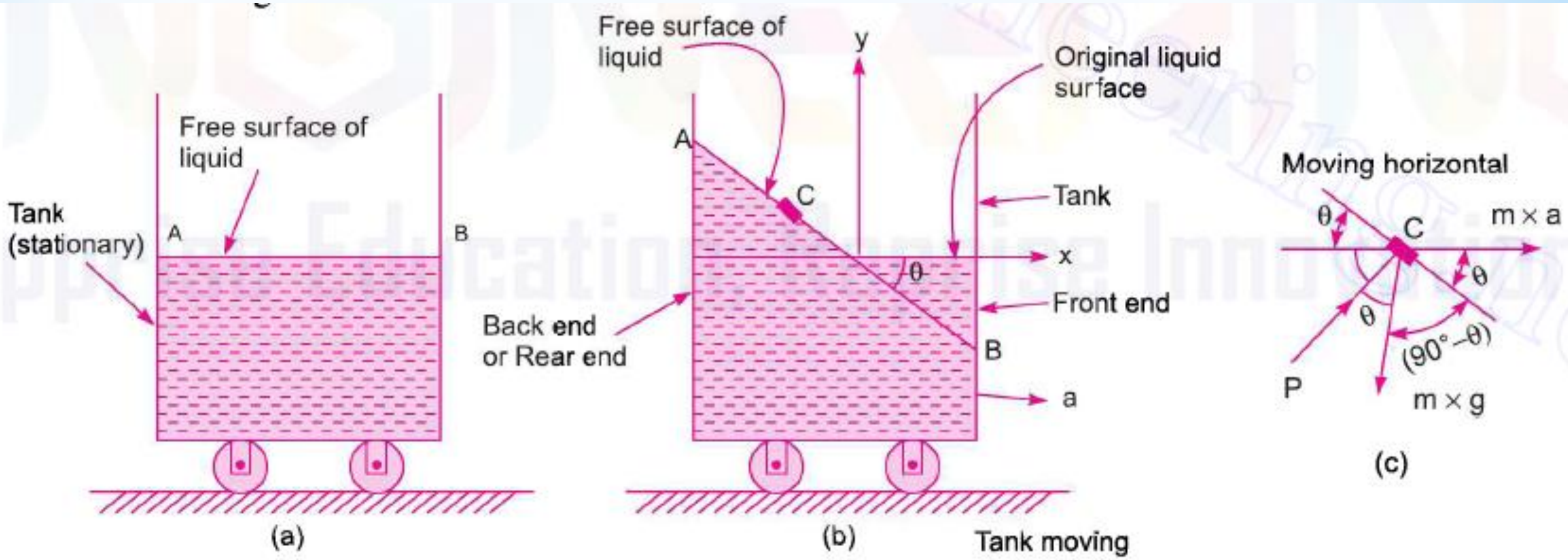
Let W = Weight of vessel including w_1

G = Centre of gravity of the vessel

B = Centre of buoyancy of the vessel

BUOYANCY AND FLOATATION

▶ 3.8 PRESSURE DISTRIBUTION IN A LIQUID SUBJECTED TO CONSTANT HORIZONTAL/VERTICAL ACCELERATION



(*i*) the pressure force P exerted by the surrounding fluid on the element C. This force is normal to the free surface.

(*ii*) the weight of the fluid element *i.e.*, $m \times g$ acting vertically downward.

(*iii*) accelerating force *i.e.*, $m \times a$ acting in horizontal direction.

BUOYANCY AND FLOATATION

Resolving the forces horizontally, we get

$$P \sin \theta + m \times a = 0$$

or $$P \sin \theta = -ma$$

Resolving the forces vertically, we get

$$P \cos \theta - mg = 0$$

or $$P \cos \theta = m \times g$$

Dividing (*i*) by (*ii*), we get

$$\tan \theta = -\frac{a}{g} \quad \left(\text{or } \frac{a}{g} \text{ Numerically}\right)$$

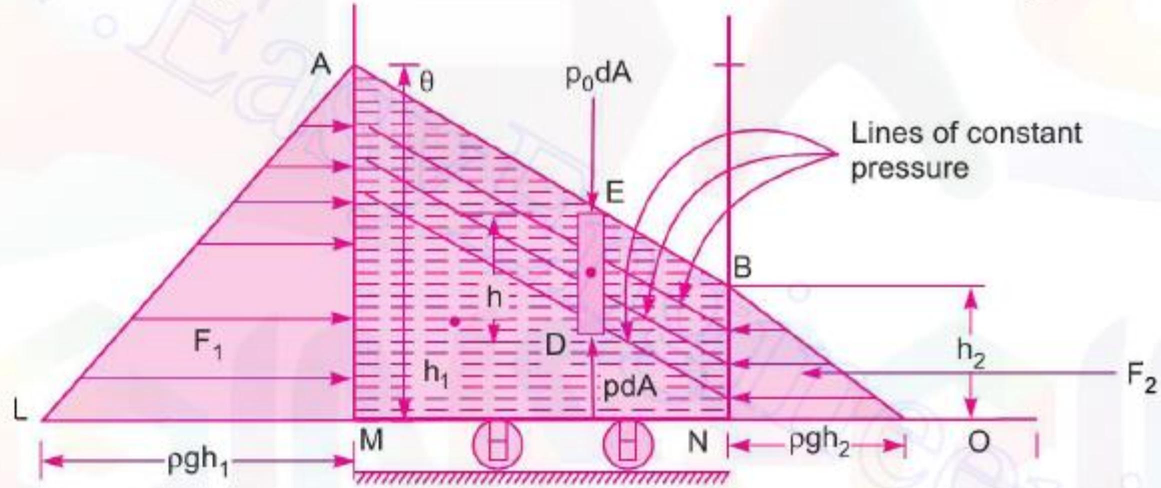


Fig. 3.44

BUOYANCY AND FLOATATION

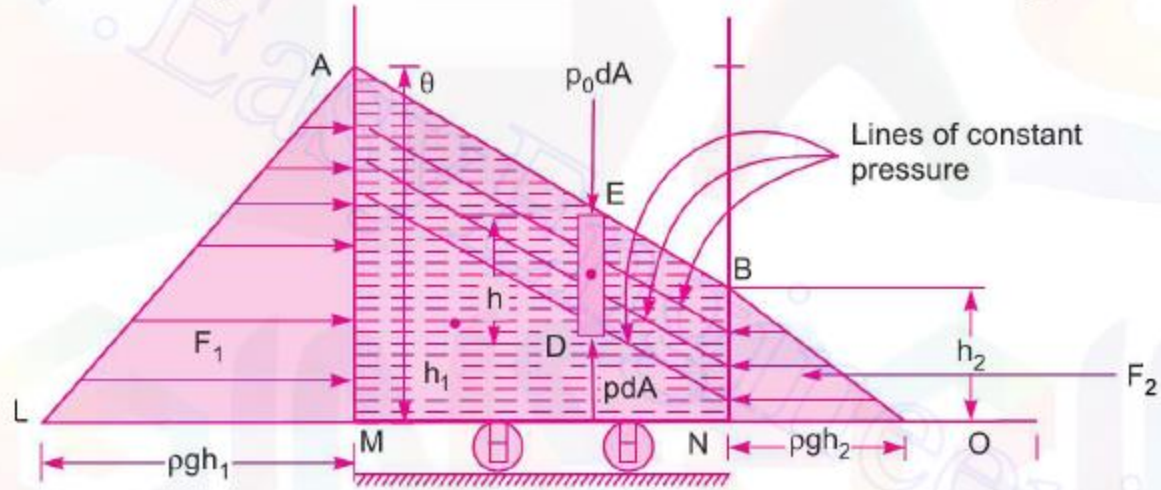


Fig. 3.44

Consider the equilibrium of the elementary prism *DE*.

The forces acting on this prism *DE* in the vertical direction are :

(*i*) the atmospheric pressure force ($p_0 \times dA$) at the top end of the prism acting downwards,

(*ii*) the weight of the element ($\rho \times g \times h \times dA$) at the C.G. of the element acting in the downward direction, and

(*iii*) the pressure force ($p \times dA$) at the bottom end of the prism acting upwards.

Since there is no vertical acceleration given to the tank, hence net force acting vertically should be zero.

BUOYANCY AND FLOATATION

$$\therefore \quad p \times dA - p_0 \times dA - \rho g h \, dA = 0$$

or $$p - p_0 - \rho g h = 0 \quad \text{or} \quad p = p_0 + \rho g h$$

or $$p - p_0 = \rho g h$$

or gauge pressure at point D is given by

$$p = \rho g h$$

or pressure head at point D, $$\frac{p}{\rho g} = h.$$

BUOYANCY AND FLOATATION

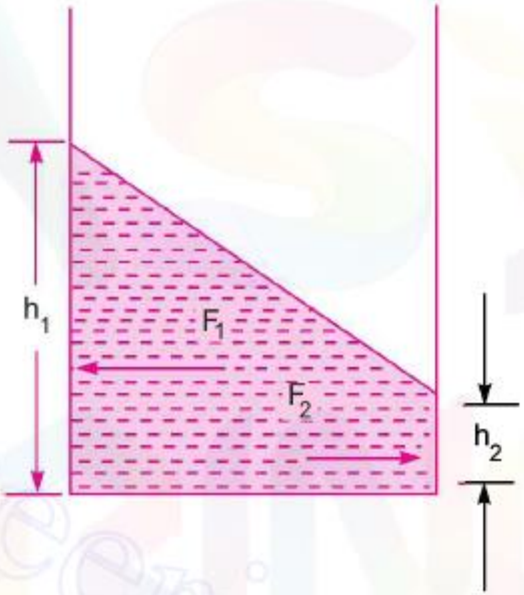


Fig. 3.44(*a*)

$$F_1 = \rho \times g \times A_1 \times \bar{h}_1, \text{ where } A_1 = h_1 \times b \text{ and } \bar{h}_1 = \frac{h_1}{2}$$

$$= \rho \times g \times (h_1 \times b) \times \frac{h_1}{2} = \frac{1}{2}\rho g . b . h_1^2$$

and

$$F_2 = \rho \times g \times A_2 \times \bar{h}_2, \text{ where } A_2 = h_2 \times b \text{ and } \bar{h}_2 = \frac{h_2}{2}$$

$$= \rho \times g \times (h_2 \times b) \times \frac{h_2}{2}$$

$$= \frac{1}{2}\rho g . b \times h_2^2.$$

It can also be proved that the difference of these two forces (*i.e.*, $F_1 - F_2$) is equal to the force required to accelerate the mass of the liquid contained in the tank *i.e.*,

$$F_1 - F_2 = M \times a$$

DYNAMICS OF FLUID FLOW

Thus dynamics of fluid flow is the study of fluid motion with the forces causing flow. The dynamic behaviour of the fluid flow is analysed by the Newton's second law of motion, which relates the acceleration with the forces. The fluid is assumed to be incompressible and non-viscous.

▶ 6.2 EQUATIONS OF MOTION

According to Newton's second law of motion, the net force F_x acting on a fluid element in the direction of x is equal to mass m of the fluid element multiplied by the acceleration a_x in the x-direction. Thus mathematically,

$$F_x = m.a_x \quad ...(6.1)$$

In the fluid flow, the following forces are present :

(*i*) F_g, gravity force.

(*ii*) F_p, the pressure force.

(*iii*) F_v, force due to viscosity.

(*iv*) F_t, force due to turbulence.

(*v*) F_c, force due to compressibility.

DYNAMICS OF FLUID FLOW

1. **Inertia Force** (F_i). It is equal to the product of mass and acceleration of the flowing fluid and acts in the direction opposite to the direction of acceleration. It is always existing in the fluid flow problems.

2. **Viscous Force** (F_v). It is equal to the product of shear stress (τ) due to viscosity and surface area of the flow. It is present in fluid flow problems where viscosity is having an important role to play.
3. **Gravity Force** (F_g). It is equal to the product of mass and acceleration due to gravity of the flowing fluid. It is present in case of open surface flow.
4. **Pressure Force** (F_p). It is equal to the product of pressure intensity and cross-sectional area of the flowing fluid. It is present in case of pipe-flow.
5. **Surface Tension Force** (F_s). It is equal to the product of surface tension and length of surface of the flowing fluid.
6. **Elastic Force** (F_e). It is equal to the product of elastic stress and area of the flowing fluid.

For a flowing fluid, the above-mentioned forces may not always be present. And also the forces, which are present in a fluid flow problem, are not of equal magnitude. There are always one or two forces which dominate the other forces. These dominating forces govern the flow of fluid.

DYNAMICS OF FLUID FLOW

Thus in equation (6.1), the net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x + (F_c)_x.$$

(*i*) If the force due to compressibility, F_c is negligible, the resulting net force

$$F_x = (F_g)_x + (F_p)_x + (F_v)_x + (F_t)_x$$

and equation of motions are called **Reynold's equations of motion.**

(*ii*) For flow, where (F_t) is negligible, the resulting equations of motion are known as **Navier-Stokes Equation.**

(*iii*) If the flow is assumed to be ideal, viscous force (F_v) is zero and equation of motions are known as **Euler's equation of motion.**

DYNAMICS OF FLUID FLOW

▶ 6.3 EULER'S EQUATION OF MOTION

Consider a stream-line in which flow is taking place in s-direction as shown in Fig. 6.1. Consider a cylindrical element of cross-section dA and length ds. The forces acting on the cylindrical element are:

1. Pressure force pdA in the direction of flow.
2. Pressure force $\left(p + \frac{\partial p}{\partial s} ds\right) dA$ opposite to the direction of flow.
3. Weight of element $\rho g dA ds$.

Let θ is the angle between the direction of flow and the line of action of the weight of element.

The resultant force on the fluid element in the direction of s must be equal to the mass of fluid element × acceleration in the direction s.

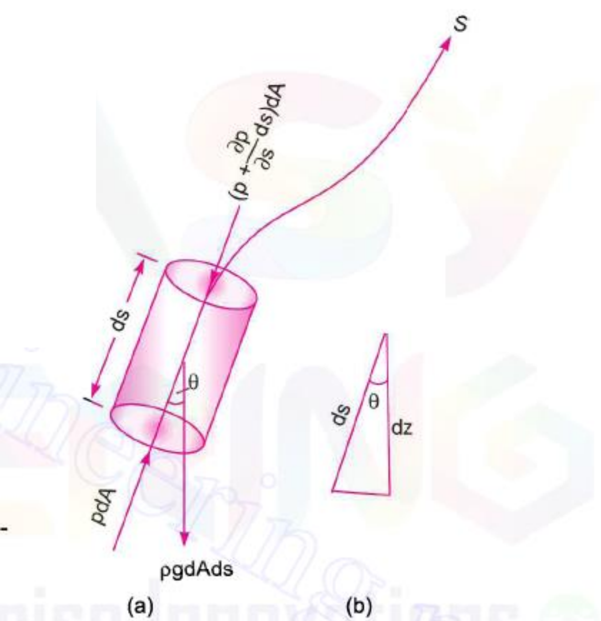


Fig. 6.1 *Forces on a fluid element.*

DYNAMICS OF FLUID FLOW

$$\therefore \quad pdA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \rho g dA ds \cos\theta$$

$$= \rho dA ds \times a_s \qquad \text{...(6.2)}$$

where a_s is the acceleration in the direction of s.

Now $\quad a_s = \frac{dv}{dt}$, where v is a function of s and t.

$$= \frac{\partial v}{\partial s}\frac{ds}{dt} + \frac{\partial v}{\partial t} = \frac{v\partial v}{\partial s} + \frac{\partial v}{\partial t} \quad \left\{\because \frac{ds}{dt} = v\right\}$$

If the flow is steady, $\frac{\partial v}{\partial t} = 0$

$$\therefore \quad a_s = \frac{v\partial v}{\partial s}$$

Substituting the value of a_s in equation (6.2) and simplifying the equation, we get

$$-\frac{\partial p}{\partial s} dsdA - \rho g\, dAds \cos\theta = \rho dAds \times \frac{v\partial v}{\partial s}$$

Fig. 6.1

Dividing by $\rho dsdA$, $-\frac{\partial p}{\rho \partial s} - g\cos\theta = \frac{v\partial v}{\partial s}$

or $$\frac{\partial p}{\rho \partial s} + g\cos\theta + v\frac{\partial v}{\partial s} = 0$$

But from Fig. 6.1 (*b*), we have $\cos\theta = \frac{dz}{ds}$

$$\therefore \quad \frac{1}{\rho}\frac{dp}{ds} + g\frac{dz}{ds} + \frac{vdv}{ds} = 0 \quad \text{or} \quad \frac{dp}{\rho} + gdz + vdv = 0$$

or $$\frac{dp}{\rho} + gdz + vdv = 0$$

Equation (6.3) is known as Euler's equation of motion.

DYNAMICS OF FLUID FLOW

▶ 6.4 BERNOULLI'S EQUATION FROM EULER'S EQUATION

Bernoulli's equation is obtained by integrating the Euler's equation of motion (6.3) as

$$\int \frac{dp}{\rho} + \int g dz + \int v dv = \text{constant}$$

If flow is incompressible, ρ is constant and

$$\therefore \qquad \frac{p}{\rho} + gz + \frac{v^2}{2} = \text{constant}$$

or

$$\frac{p}{\rho g} + z + \frac{v^2}{2g} = \text{constant}$$

or

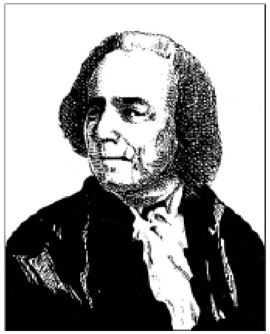
$$\frac{p}{\rho g} + \frac{v^2}{2g} + z = \text{constant} \quad \ldots$$

Equation (6.4) is a Bernoulli's equation in which

$\frac{p}{\rho g}$ = pressure energy per unit weight of fluid or pressure head.

$v^2/2g$ = kinetic energy per unit weight or kinetic head.

z = potential energy per unit weight or potential head.



Leonhard Euler (1707–83)
Mathematician born near Basle in Switzerland. A pupil of Johann Bernoulli and a close friend of Daniel Bernoulli. Contributed enormously to the mathematical development of Newtonian mechanics, while formulating the equations of motion of a perfect fluid and solid. Lost his sight in one eye and then both eyes, as a result of a disease, but still continued his research.

▶ 6.5 ASSUMPTIONS

The following are the assumptions made in the derivation of Bernoulli's equation :

(*i*) The fluid is ideal, *i.e.*, viscosity is zero (*ii*) The flow is steady

(*iii*) The flow is incompressible (*iv*) The flow is irrotational.

DYNAMICS OF FLUID FLOW

Daniel Bernoulli (1700–82)
Mathematician born in Groningen in the Netherlands. A good friend of Euler. Made efforts to popularise the law of fluid motion, while tackling various novel problems in fluid statics and dynamics. Originated the Latin word *hydro-dynamica*, meaning fluid dynamics.

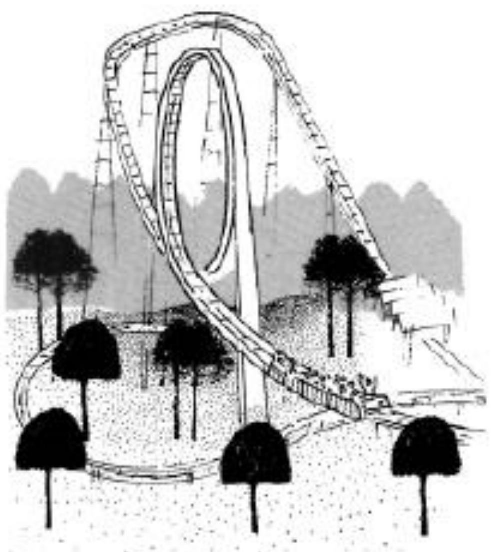
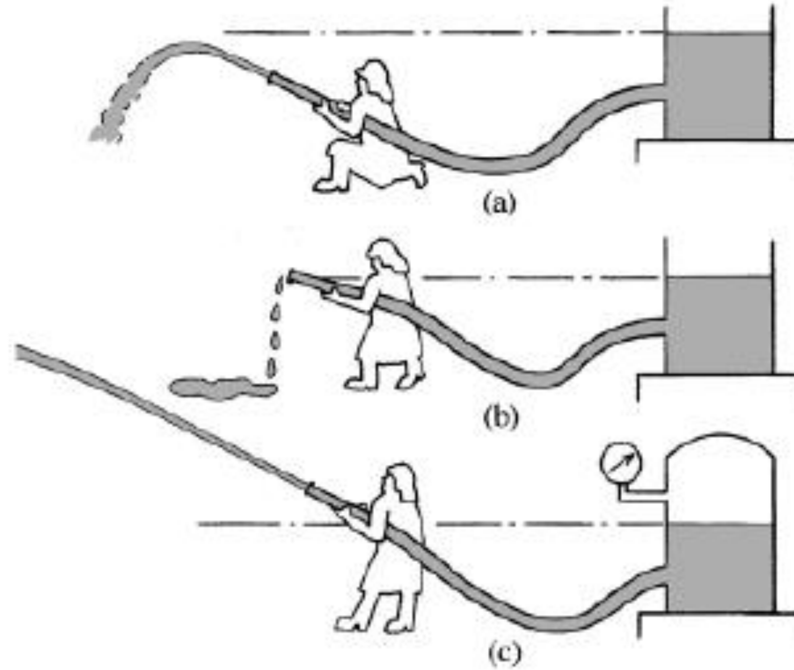


Fig. 5.2 Movement of roller-coaster



. 5.3 Conservation of fluid energy

DYNAMICS OF FLUID FLOW

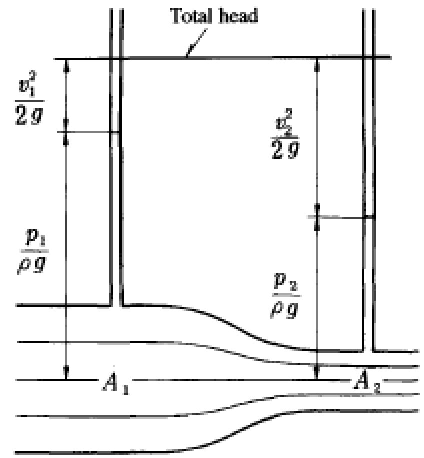


Fig. 5.6 Exchange between pressure head and velocity head

$$\frac{v_1^2}{2} + \frac{p_1}{\rho} + z_1 = \frac{v_2^2}{2} + \frac{p_2}{\rho} + z_2 + h_2 = \frac{v_3^2}{2} + \frac{p_3}{\rho} + z_3 + h_3$$

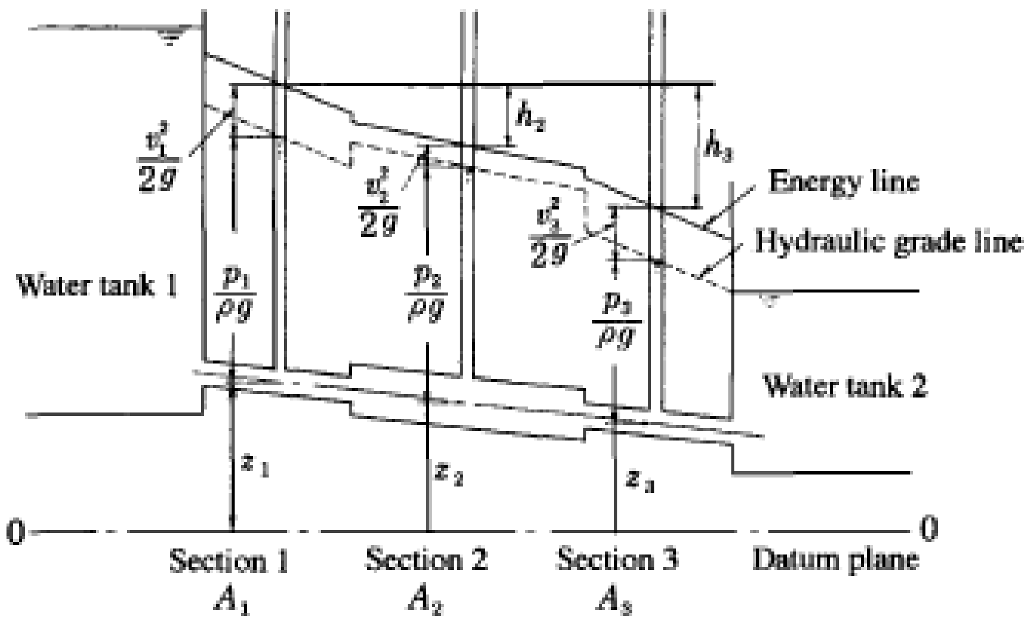


Fig. 5.7 Hydraulic grade line and energy line

DYNAMICS OF FLUID FLOW

Problem 6.2 *A pipe, through which water is flowing, is having diameters, 20 cm and 10 cm at the cross-sections 1 and 2 respectively. The velocity of water at section 1 is given 4.0 m/s. Find the velocity head at sections 1 and 2 and also rate of discharge.*

Solution. Given :

$D_1 = 20 \text{ cm} = 0.2 \text{ m}$

∴ Area, $A_1 = \frac{\pi}{4} D_1^2 = \frac{\pi}{4} (.2)^2 = 0.0314 \text{ m}^2$

$V_1 = 4.0 \text{ m/s}$

$D_2 = 0.1 \text{ m}$

∴ $A_2 = \frac{\pi}{4} (.1)^2 = .00785 \text{ m}^2$

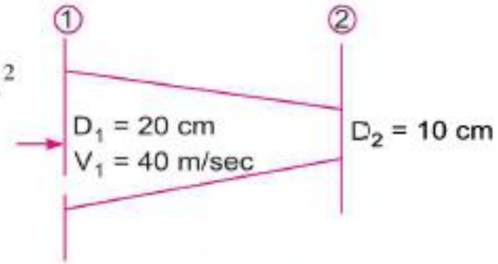


Fig. 6.2

(*i*) Velocity head at section 1

$$= \frac{V_1^2}{2g} = \frac{4.0 \times 4.0}{2 \times 9.81} = \mathbf{0.815 \text{ m. Ans.}}$$

(*ii*) Velocity head at section 2 = $V_2^2/2g$

To find V_2, apply continuity equation at 1 and 2

∴ $A_1V_1 = A_2V_2$ or $V_2 = \frac{A_1V_1}{A_2} = \frac{.0314}{.00785} \times 4.0 = 16.0 \text{ m/s}$

∴ Velocity head at section 2 $= \frac{V_2^2}{2g} = \frac{16.0 \times 16.0}{2 \times 9.81} = \mathbf{83.047 \text{ m. Ans.}}$

(*iii*) Rate of discharge $= A_1V_1$ or A_2V_2

$= 0.0314 \times 4.0 = 0.1256 \text{ m}^3/\text{s}$

$=$ **125.6 litres/s. Ans.** $\{\because 1 \text{ m}^3 = 1000 \text{ litres}\}$

DYNAMICS OF FLUID FLOW

▶ 6.6 BERNOULLI'S EQUATION FOR REAL FLUID

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_L$$

where h_L is loss of energy between points 1 and 2.

▶ 6.7 PRACTICAL APPLICATIONS OF BERNOULLI'S EQUATION

Bernoulli's equation is applied in all problems of incompressible fluid flow where energy considerations are involved. But we shall consider its application to the following measuring devices :

1. Venturimeter.
2. Orifice meter.
3. Pitot-tube.

DYNAMICS OF FLUID FLOW

Types of venturimeters:

Venturimeters may be *classified* as follows:

1. Horizontal venturimeters.
2. Vertical venturimeters.
3. Inclined venturimeters.

Giovanni Battista Venturi (1746–1822)
Italian physicist. After experiencing life as a priest, teacher and auditor, finally became a professor of experimental physics. Studied the effects of eddies and the flow rates at various forms of mouthpieces fitted to an orifice, and clarified the basic principles of the Venturi tube and the hydraulic jump in an open water channel.

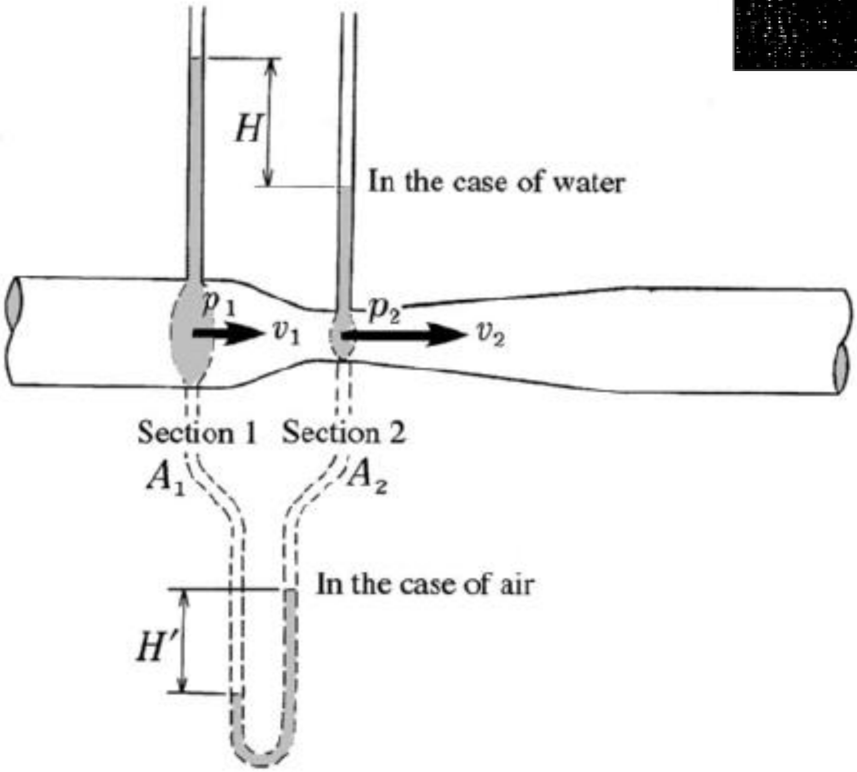


Fig. 5.8 Venturi tube

DYNAMICS OF FLUID FLOW

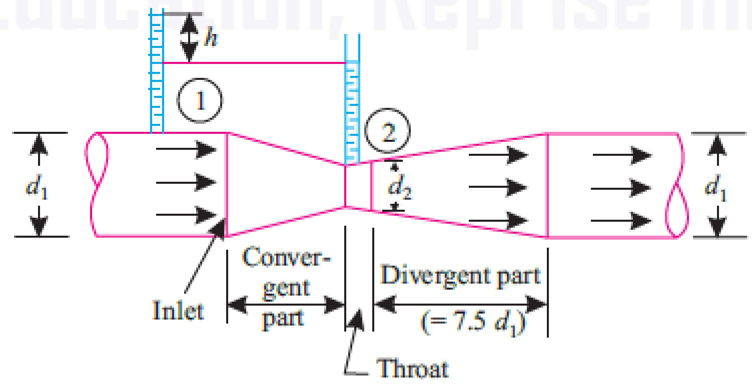
6.6.1.1. Horizontal venturimeters

A venturimeter consists of the following *three* parts:

(*i*) A short converging part,

(*ii*) Throat, and

(*iii*) Diverging part.



(Throat ratio $\frac{d_2}{d_1}$ varies $\frac{1}{4}$ to $\frac{3}{4}$)

or, $$Q = \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh} \quad ...(6.5)$$

or, $$Q = C\sqrt{h}$$

where, C = constant of venturimeter

$$= \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}}\sqrt{2g}$$

Eqn. (6.5) gives the discharge under ideal conditions and is called *theoretical discharge*. Actual discharger (Q_{act}) which is less than the theoretical discharge (Q_{th}) is given by:

$$Q_{act} = C_d \times \frac{A_1 A_2}{\sqrt{A_1^2 - A_2^2}} \times \sqrt{2gh} \quad ...(6.6)$$

where, C_d = *Co-efficient of venturimeter* (or co-efficient of discharge) and its value is *less than unity* (varies between 0.96 and 0.98)

- *Due to variation of C_d venturimeters are not suitable for very low velocities.*

DYNAMICS OF FLUID FLOW

Value of 'h' given by differential U-tube manometer

Case I. Let the differential manometer contains a liquid which is heavier than the liquid flowing through the pipe. Let

S_h = Sp. gravity of the heavier liquid

S_o = Sp. gravity of the liquid flowing through pipe

x = Difference of the heavier liquid column in U-tube

Then

$$h = x\left[\frac{S_h}{S_o} - 1\right] \qquad \text{...(6.9)}$$

Case II. If the differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given by

$$h = x\left[1 - \frac{S_l}{S_o}\right]$$

where S_l = Sp. gr. of lighter liquid in U-tube

S_o = Sp. gr. of fluid flowing through pipe

x = Difference of the lighter liquid columns in U-tube.

DYNAMICS OF FLUID FLOW

Case III. Inclined Venturimeter with Differential U-tube manometer. The above two cases are given for a horizontal venturimeter. This case is related to inclined venturimeter having differential U-tube manometer. Let the differential manometer contains heavier liquid then h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[\frac{S_h}{S_o} - 1\right] \qquad \text{...(6.11)}$$

Case IV. Similarly, for inclined venturimeter in which differential manometer contains a liquid which is lighter than the liquid flowing through the pipe, the value of h is given as

$$h = \left(\frac{p_1}{\rho g} + z_1\right) - \left(\frac{p_2}{\rho g} + z_2\right) = x\left[1 - \frac{S_l}{S_o}\right] \qquad \text{...(6.12)}$$

DYNAMICS OF FLUID FLOW

6.7.2 Orifice Meter or Orifice Plate.

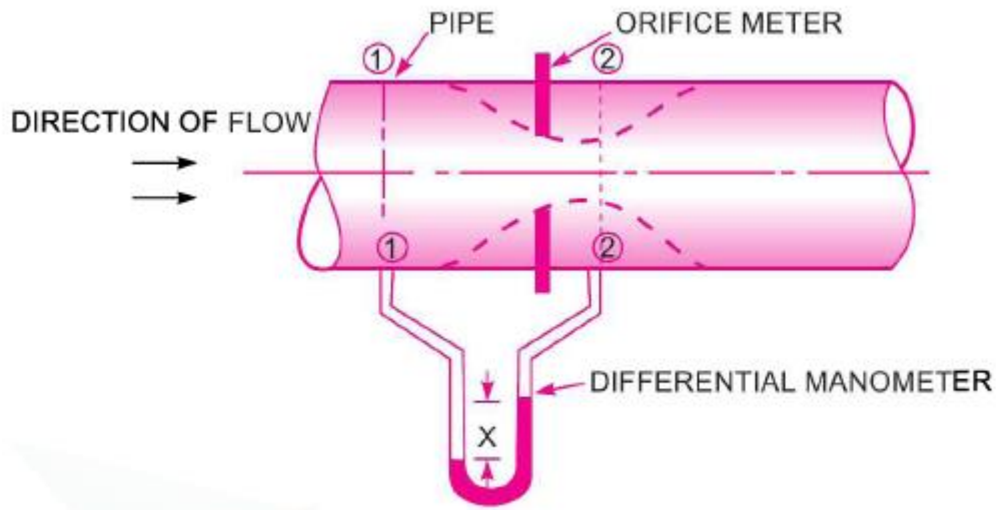


Fig. 6.12. *Orifice meter.*

$$Q = a_0 \times C_d \frac{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}} \times \frac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2 C_c^2}}$$

$$= \frac{C_d a_0 \sqrt{2gh}}{\sqrt{1 - \left(\frac{a_0}{a_1}\right)^2}} = \frac{C_d a_0 a_1 \sqrt{2gh}}{\sqrt{a_1^2 - a_0^2}}. \quad \ldots$$

where C_d = Co-efficient of discharge for orifice meter.

The co-efficient of discharge for orifice meter is much smaller than that for a venturimeter.

DYNAMICS OF FLUID FLOW

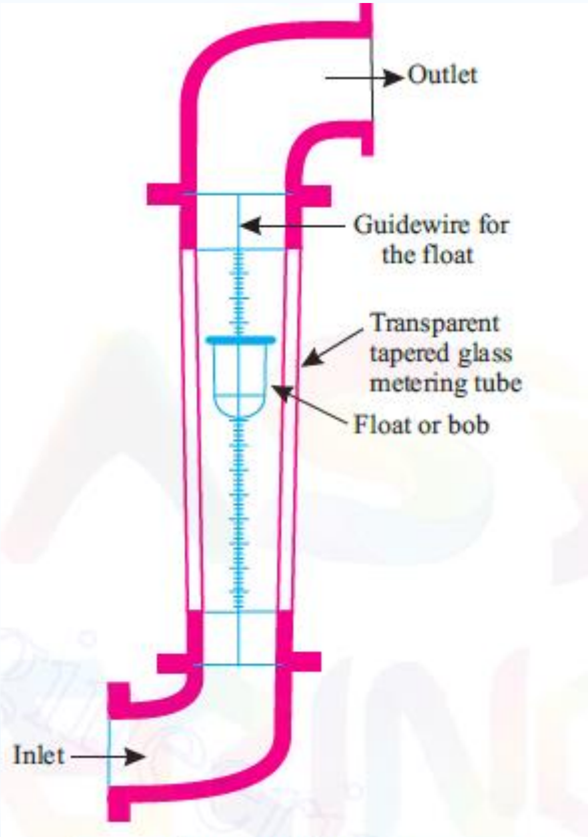


Fig. 6.36. Rotameter.

DYNAMICS OF FLUID FLOW

6.7.3 Pitot-tube.

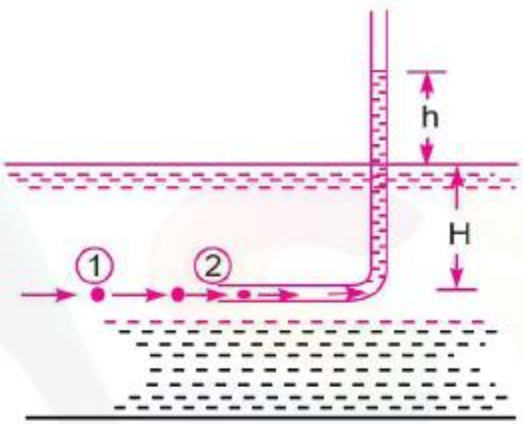


Fig. 6.13 *Pitot-tube.*

Henry de Pitot (1695–1771)

Born in Aramon in France. Studied mathematics and physics in Paris. As a civil engineer, undertook the drainage of marshy lands, construction of bridges and city water systems, and flood countermeasures. His books cover structures, land survey, astronomy, mathematics, sanitary equipment and theoretical ship steering in addition to hydraulics. The famous Pitot tube was announced in 1732 as a device to measure flow velocity.

Let p_1 = intensity of pressure at point (1)

v_1 = velocity of flow at (1)

p_2 = pressure at point (2)

v_2 = velocity at point (2), which is zero

H = depth of tube in the liquid

h = rise of liquid in the tube above the free surface.

Applying Bernoulli's equation at points (1) and (2), we get

DYNAMICS OF FLUID FLOW

DYNAMICS OF FLUID FLOW

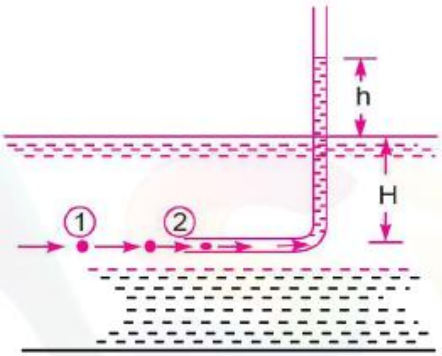


Fig. 6.13 *Pitot-tube.*

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

But $z_1 = z_2$ as points (1) and (2) are on the same line and $v_2 = 0$.

$$\frac{p_1}{\rho g} = \text{pressure head at (1)} = H$$

$$\frac{p_2}{\rho g} = \text{pressure head at (2)} = (h + H)$$

Substituting these values, we get

$$\therefore \quad H + \frac{v_1^2}{2g} = (h + H) \quad \therefore \quad h = \frac{v_1^2}{2g} \quad \text{or} \quad v_1 = \sqrt{2gh}$$

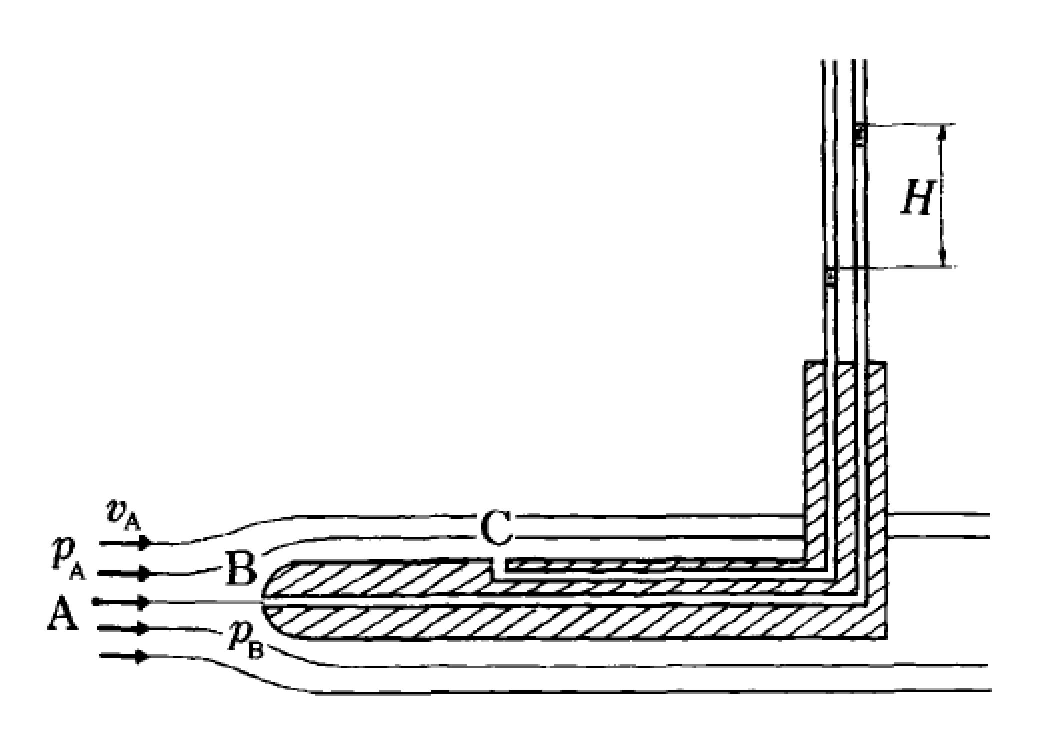
This is theoretical velocity. Actual velocity is given by

$$(v_1)_{act} = C_v \sqrt{2gh}$$

where C_v = Co-efficient of pitot-tube

$$\therefore \quad \text{Velocity at any point} \quad v = C_v \sqrt{2gh}$$

DYNAMICS OF FLUID FLOW



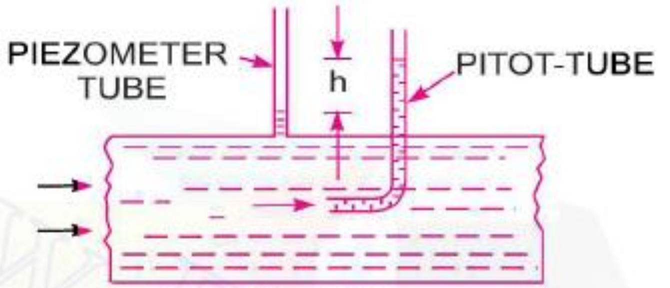


Fig. 6.14

DYNAMICS OF FLUID FLOW

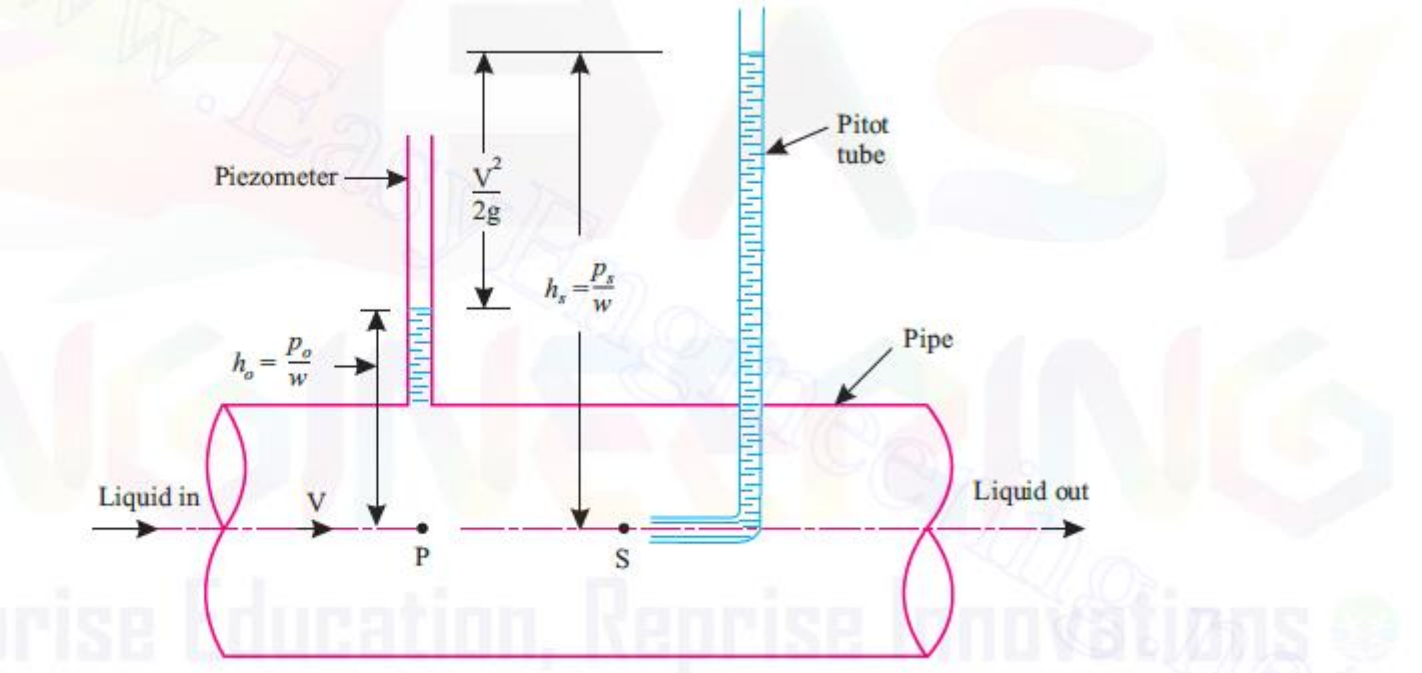


Fig. 6.38. Pitot tube.

$$\frac{p_0}{w} + \frac{V^2}{2g} = \frac{p_s}{w} \quad \text{or} \quad h_0 + \frac{V^2}{2g} = h_s$$

$$V = \sqrt{2g\,(h_s - h_0)} \quad \text{or} \quad \sqrt{2g\,\Delta h} \qquad \ldots(1)$$

p_0 = Pressure at point '*P*', *i.e.* static pressure,

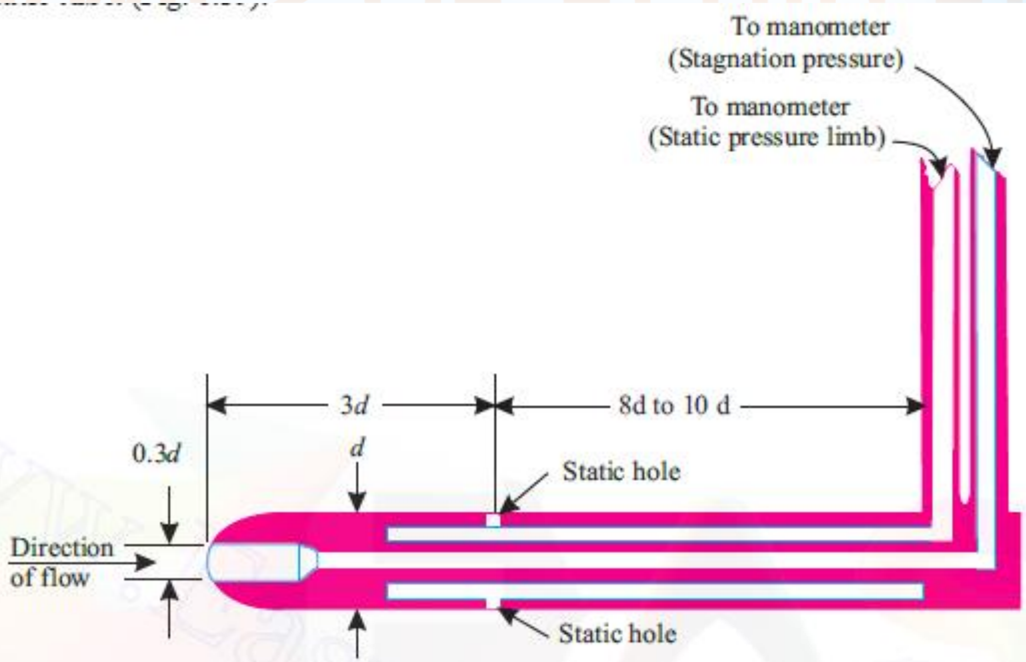
V = Velocity at point '*P*', *i.e.* free flow velocity,

p_s = Stagnation pressure at point '*S*', and

Δh = Dynamic pressure

= Difference between stagnation pressure head (h_s) and static pressure head (h_0).

DYNAMICS OF FLUID FLOW



If y is the manometric difference, then

$$\Delta h = y\left(\frac{S_m}{S} - 1\right)$$

where, S_m = Specific gravity of manometric liquid, and

S = Specific gravity of the liquid flowing through the pipe.

$$V = C\sqrt{2g\Delta h} \qquad ...(2)$$

where, C = A connective coefficient which takes into account the effect of stem and bent leg.

The most commonly used form of Pitot static tube known as the Prandle-Pitot-tube is so designed that the effect of stem and bent leg cancel each other, *i.e.*, $C = 1$.

DYNAMICS OF FLUID FLOW

IMPULSE-MOMENTUM EQUATION

"The net force acting on a mass of fluid is equal to change in momentum of flow per unit time in that direction".

As per Newton's second law of motion,

$$F = ma$$

where, m = Mass of fluid,
F = Force acting on the fluid, and
a = Acceleration (acting in the same direction as F).

But acceleration, $a = \frac{dv}{dt}$

$$\therefore \quad F = m.\frac{dv}{dt} = \frac{d(mv)}{dt}$$

('m' is taken inside the differential, being constant)

This equation is known as **momentum principle.** It can also be written as:

$$F.dt = d\,(mv)$$

This equation is known as **Impulse-momentum equation**. It may be stated as follows:

"The impulse of a force F acting on a fluid mass 'm' in a short interval of time dt is equal to the change of momentum d(mv) in direction of force".

The impulse-momentum equations are often called simply *momentum equations*.

DYNAMICS OF FLUID FLOW

Force exerted by a flowing fluid on a pipe bend

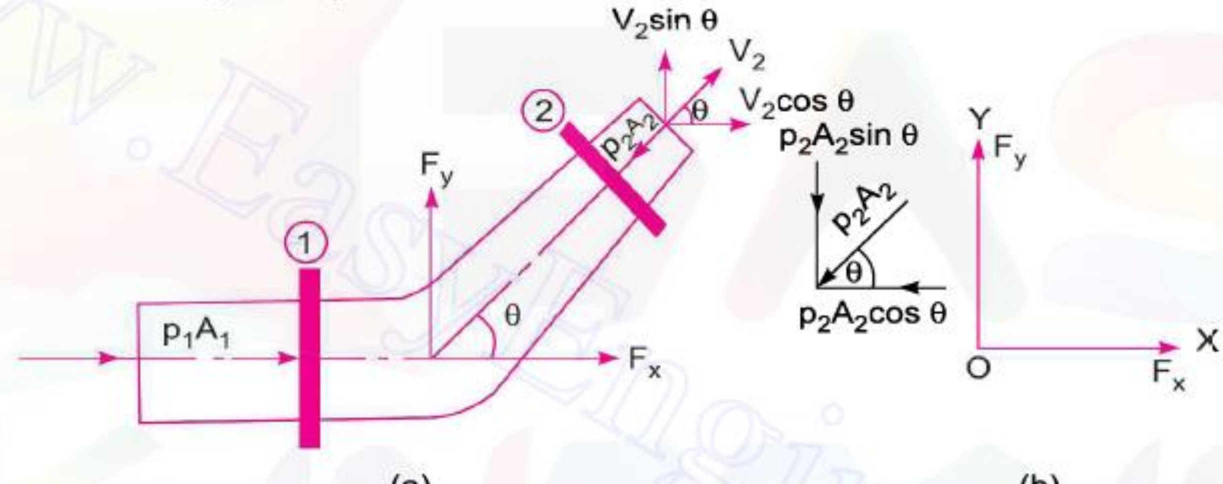


Fig. 6.18 *Forces on bend.*

$$\therefore \quad F_x = \rho Q\,(V_1 - V_2 \cos\theta) + p_1A_1 - p_2A_2 \cos\theta$$

$$\therefore \quad F_y = \rho Q\,(-V_2 \sin\theta) - p_2A_2 \sin\theta$$

Now the resultant force (F_R) acting on the bend

$$= \sqrt{F_x^2 + F_y^2}$$

And the angle made by the resultant force with horizontal direction is given by

$$\tan\theta = \frac{F_y}{F_x}$$

DYNAMICS OF FLUID FLOW

MOMENT OF MOMENTUM EQUATION

Moment of momentum equation is derived from *moment of momentum principle* which states as follows:

"The resulting torque acting on a rotating fluid is equal to the rate of change of moment of momentum".

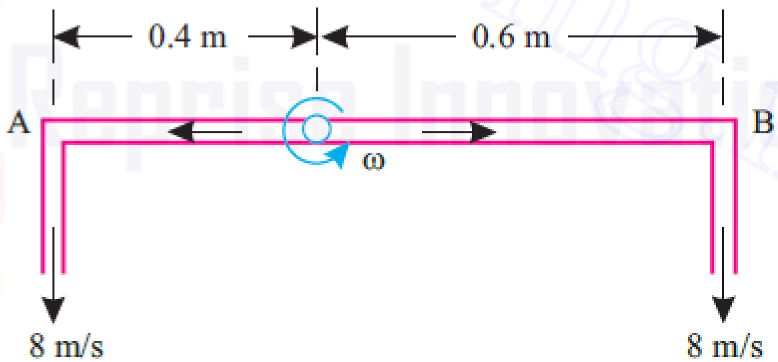
Q = Steady rate of flow of fluid,

ρ = Density of fluid,

V_1 = Velocity of fluid at section 1,

r_1 = Radius of curvature at section 1, and

V_2 and r_2 = Velocity and radius of curvature at section 2.



Momentum of fluid at section 1 = Mass × velocity = $\rho Q \times V_1$

∴ Moment of momentum per second of fluid at section 1 = $\rho Q \times V_1 \times r_1$

Similarly, moment of momentum per second of fluid at section 2 = $\rho Q \times V_2 \times r_2$

∴ Rate of change of moment of momentum = $\rho Q V_2 r_2 - \rho Q V_1 r_1 = \rho Q (V_2 r_2 - V_1 r_1)$

According to the moment of momentum principle,

Resultant torque = Rate of change of moment of momentum

$$T = \rho Q (V_2 r_2 - V_1 r_1)$$

Eqn. (6.31) is known as *moment of momentum equation*. This equation is used:

FLOW THROUGH ORIFICES

8.1. INTRODUCTION

An **orifice** *is an opening in the wall or base of a vessel through which the fluid flows.* The *top edge* of the orifice is always below *the free surface* (If the free surface is below the top edge of the orifice, becomes a *weir*)

A **mouthpiece** *is an attachmnent in the form of a small tube or pipe fixed to the orifice (the length of pipe extension is usually 2 to 3 times the orifice diameter) and is used to increase the amount of dischrge.*

- Orifices as well as mouthpieces are used to *measure the discharge.*

FLOW THROUGH ORIFICES

8.2. CLASSIFICATION OF ORIFICES

The orifices are *classified* as follows

1. According to size:

(*i*) Small orifice (*ii*) Large orifice.

An orifice is termed *small* when its dimensions are small compared to the head causing flow. The velocity does not vary appreciably from top to the bottom edge of the orifice and is assumed to be uniform.

The orifice is *large* if the dimensions are comparable with the head causing flow. The variation in the velocity from the top to the bottom edge is considerable.

2. According to shape

(*i*) Circular orifice (*ii*) Rectangular orifice

(*iii*) Square orifice (*iv*) Triangular orifice.

3. Shape of upstream edge

(*i*) Sharp-edged orifice

(*ii*) Bell-mouthed orifice.

4. According to discharge conditions

(*i*) Free discharge orifices

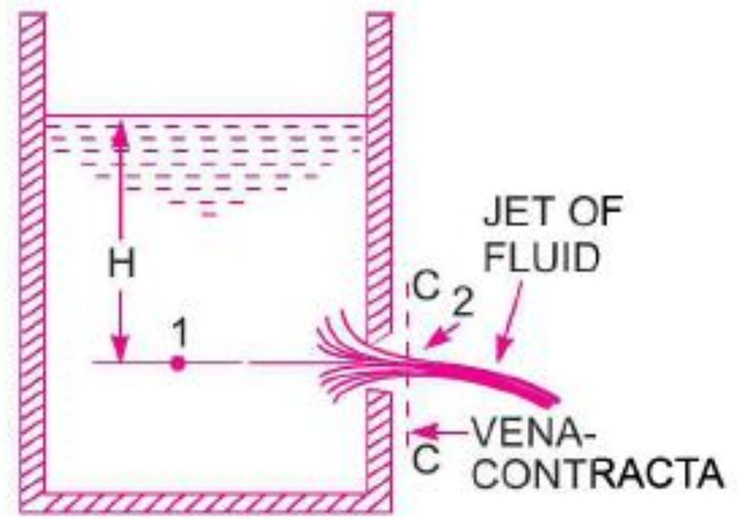
(*ii*) Drowned or submerged orifices

(*a*) Fully submereged

(*b*) Partially submerged.

Note. An orifice or a mouthpiece is said to be discharging *free* when it discharges into *atmosphere*. It is said to be *submerged* when it discharges into *another liquid*.

FLOW THROUGH ORIFICES



ig. 7.1 *Tank with an orifice.*

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2$$

$$z_1 = z_2$$

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

$$\frac{p_1}{\rho g} = H$$

$$\frac{p_2}{\rho g} = 0 \quad \text{(atmospheric pressure)}$$

v_1 is very small in comparison to v_2 as area of tank is very large as compared to the area of the jet of liquid.

$$\therefore \quad H + 0 = 0 + \frac{v_2^2}{2g}$$

$$\therefore \quad v_2 = \sqrt{2gH} \qquad \text{...(7.1)}$$

This is theoretical velocity. Actual velocity will be less than this value.

FLOW THROUGH ORIFICES

▶ 7.4 HYDRAULIC CO-EFFICIENTS

The hydraulic co-efficients are

1. Co-efficient of velocity, C_v
2. Co-efficient of contraction, C_c
3. Co-efficient of discharge, C_d.

7.4.1 Co-efficient of Velocity (C_v). It is defined as the ratio between the actual velocity of a jet of liquid at vena-contracta and the theoretical velocity of jet. It is denoted by C_v and mathematically, C_v is given as

$$C_v = \frac{\text{Actual velocity of jet at vena-contracta}}{\text{Theoretical velocity}}$$

$$= \frac{V}{\sqrt{2gH}}, \text{ where } V = \text{actual velocity, } \sqrt{2gH} = \text{Theoretical velocity} \quad \text{...(7.2)}$$

The value of C_v varies from 0.95 to 0.99 for different orifices, depending on the shape, size of the orifice and on the head under which flow takes place. Generally the value of $C_v = 0.98$ is taken for sharp-edged orifices.

FLOW THROUGH ORIFICES

7.4.2 Co-efficient of Contraction (C_c). It is defined as the ratio of the area of the jet at vena-contracta to the area of the orifice. It is denoted by C_c.

Let a = area of orifice and

a_c = area of jet at vena-contracta.

Then
$$C_c = \frac{\text{area of jet at vena-contracta}}{\text{area of orifice}}$$

$$= \frac{a_c}{a} \quad \text{...(7.3)}$$

The value of C_c varies from 0.61 to 0.69 depending on shape and size of the orifice and head of liquid under which flow takes place. In general, the value of C_c may be taken as 0.64.

7.4.3 Co-efficient of Discharge (C_d). It is defined as the ratio of the actual discharge from an orifice to the theoretical discharge from the orifice. It is denoted by C_d. If Q is actual discharge and Q_{th} is the theoretical discharge then mathematically, C_d is given as

$$C_d = \frac{Q}{Q_{th}} = \frac{\text{Actual velocity} \times \text{Actual area}}{\text{Theoretical velocity} \times \text{Theoretical area}}$$

$$= \frac{\text{Actual velocity}}{\text{Theoretical velocity}} \times \frac{\text{Actual area}}{\text{Theoretical area}}$$

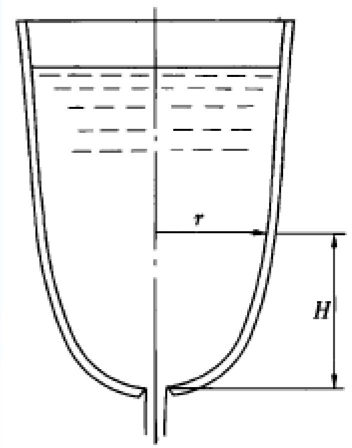
$$\therefore \quad C_d = C_v \times C_c \quad \text{...(7.4)}$$

The value of C_d varies from 0.61 to 0.65. For general purpose the value of C_d is taken as 0.62.

FLOW THROUGH ORIFICES

TIME REQUIRED FOR EMPTYING A TANK THROUGH AN ORIFICE AT ITS BOTTOM

Egyptian water clock 3400 years old (London Science Museum)



FLOW THROUGH ORIFICES

TIME REQUIRED FOR EMPTYING A TANK THROUGH AN ORIFICE AT ITS BOTTOM

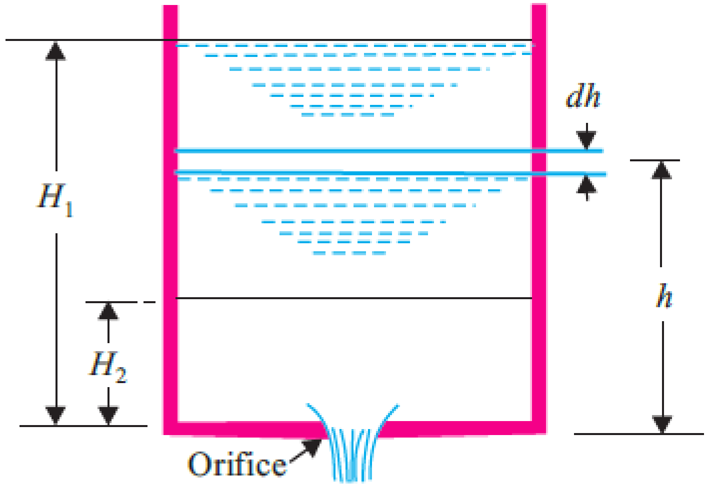


Fig. 8.13 Tank with an orifice at its bottom.

Let, A = Cross-sectional area of the tank,

a = Area of the orifice,

H_1 = Initial height of liquid,

H_2 = Final height of liquid

T = Time in seconds, required to bring the level from H_1 to H_2

Let at some instant the height of the liquid be h above the orifice and let the liquid surface fall by an amount dh after a small interval for time dt.

Then, volume of the liquid that has passed the tank in time dt,

$$dq = -A \cdot dh \qquad \ldots(i)$$

FLOW THROUGH ORIFICES

or
$$T = \frac{-A}{C_d \, . a \, . \sqrt{2g}} \left[\frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_{H_1}^{H_2} = \frac{-A}{C_d \, . a \, . \sqrt{2g}} \left[\frac{\sqrt{h}}{\frac{1}{2}} \right]_{H_1}^{H_2}$$

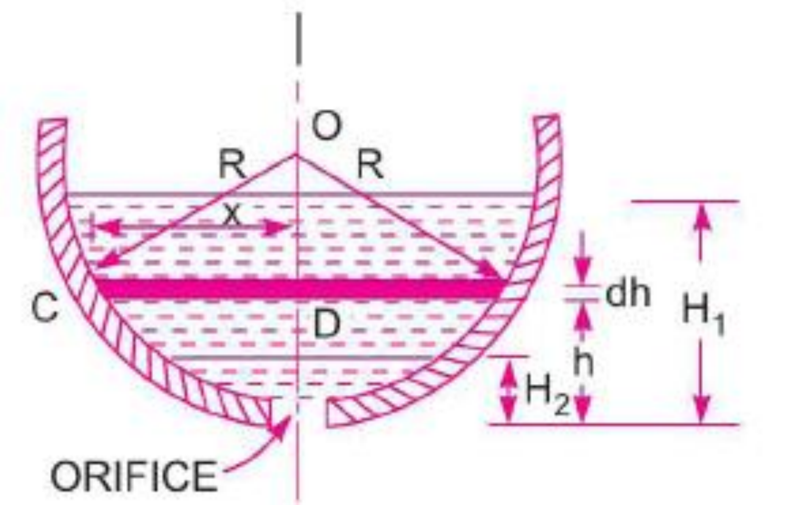
$$= \frac{-2A}{C_d \, . a \, . \sqrt{2g}} \left[\sqrt{H_2} - \sqrt{H_1} \right] = \frac{2A \left[\sqrt{H_1} - \sqrt{H_2} \right]}{C_d \, . a \, . \sqrt{2g}} \qquad \text{...(7.11)}$$

For emptying the tank completely, $H_2 = 0$ and hence

$$T = \frac{2A \sqrt{H_1}}{C_d \, . a \, . \sqrt{2g}}. \qquad \text{...(7.12)}$$

FLOW THROUGH ORIFICES

TIME OF EMPTYING A HEMISPHERICAL TANK



$$= \frac{\pi}{C_d \times a \times \sqrt{2g}}\left[\frac{4}{3}R\left(H_1^{3/2} - H_2^{3/2}\right) - \frac{2}{5}\left(H_1^{5/2} - H_2^{5/2}\right)\right] \quad ...(7.13)$$

For completely emptying the tank, $H_2 = 0$ and hence

$$T = \frac{\pi}{C_d \,.\, a \,.\, \sqrt{2g}}\left[\frac{4}{3}RH_1^{3/2} - \frac{2}{5}H_1^{5/2}\right]. \quad ...(7.14)$$

DIMENSIONAL AND MODEL ANALYSIS

S. No.	Physical Quantity	Symbol	Dimensions
	(*a*) **Fundamental**		
1.	Length	L	L
2.	Mass	M	M
3.	Time	T	T

▶ 12.2 SECONDARY OR DERIVED QUANTITIES

S.No.	Physical Quantity	Symbol	Dimensions
	(*b*) **Geometric**		
4.	Area	A	L^2
5.	Volume	$\forall$	L^3
	(*c*) **Kinematic Quantities**		
6.	Velocity	v	LT^{-1}
7.	Angular Velocity	ω	T^{-1}
8.	Acceleration	a	LT^{-2}
9.	Angular Acceleration	α	T^{-2}
10.	Discharge	Q	L^3T^{-1}
11.	Acceleration due to Gravity	g	LT^{-2}
12.	Kinematic Viscosity	ν	L^2T^{-1}

DIMENSIONAL AND MODEL ANALYSIS

	(*d*) **Dynamic Quantities**		
13.	Force	F	MLT^{-2}
14.	Weight	W	MLT^{-2}
15.	Density	ρ	ML^{-3}
16.	Specific Weight	w	$ML^{-2}T^{-2}$
17.	Dynamic Viscosity	μ	$ML^{-1}T^{-1}$
18.	Pressure Intensity	p	$ML^{-1}T^{-2}$
19.	Modulus of Elasticity	$\begin{cases} K \\ E \end{cases}$	$ML^{-1}T^{-2}$
20.	Surface Tension	σ	MT^{-2}
21.	Shear Stress	τ	$ML^{-1}T^{-2}$
22.	Work, Energy	W or E	$ML^{2}T^{-2}$
23.	Power	P	$ML^{2}T^{-3}$
24.	Torque	T	$ML^{2}T^{-2}$
25.	Momentum	M	MLT^{-1}

DIMENSIONAL AND MODEL ANALYSIS

Solution. (*i*) Angular velocity $= \dfrac{\text{Angle covered in radians}}{\text{Time}} = \dfrac{1}{T} = T^{-1}$.

(*ii*) Angular acceleration $= \text{rad/sec}^2 = \dfrac{\text{rad}}{T^2} = \dfrac{1}{T^2} = T^{-2}$.

(*iii*) Discharge $= \text{Area} \times \text{Velocity} = L^2 \times \dfrac{L}{T} = \dfrac{L^3}{T} = L^3T^{-1}$.

(*iv*) Kinematic viscosity (ν) $= \dfrac{\mu}{\rho}$, where μ is given by $\tau = \mu \dfrac{\partial u}{\partial y}$

$$\therefore \quad \mu = \frac{\tau}{\frac{\partial u}{\partial y}} = \frac{\text{Shear Stress}}{\frac{L}{T} \times \frac{1}{L}} = \frac{\frac{\text{Force}}{\text{Area}}}{\frac{1}{T}}$$

$$= \frac{\text{Mass} \times \text{Acceleration}}{\text{Area} \times \text{Time}} = \frac{M \times \frac{L}{T^2}}{L^2 \times \frac{1}{T}} = \frac{ML}{L^2T^2 \times \frac{1}{T}} = \frac{M}{LT} = ML^{-1}T^{-1}$$

and $\rho = \dfrac{\text{Mass}}{\text{Volume}} = \dfrac{M}{L^3} = ML^{-3}$

$\therefore$ Kinematic viscosity $(\nu) = \dfrac{\mu}{\rho} = \dfrac{ML^{-1}T^{-1}}{ML^{-3}} = L^2T^{-1}$.

(*v*) Force $= \text{Mass} \times \text{Acceleration} = M \times \dfrac{\text{Length}}{(\text{Time})^2} = \dfrac{ML}{T^2} = MLT^{-2}$.

(*vi*) Specific weight $= \dfrac{\text{Weight}}{\text{Volume}} = \dfrac{\text{Force}}{\text{Volume}} = \dfrac{MLT^{-2}}{L^3} = ML^{-2}T^{-2}$.

(*vii*) Dynamic viscosity, μ is derived in (*iv*) as $\mu = ML^{-1}T^{-1}$.

DIMENSIONAL AND MODEL ANALYSIS

▶ 12.3 DIMENSIONAL HOMOGENEITY

Dimensional homogeneity means the dimensions of each terms in an equation on both sides are equal. Thus if the dimensions of each term on both sides of an equation are the same the equation is known as dimensionally homogeneous equation. The powers of fundamental dimensions (*i.e.*, L, M, T) on both sides of the equation will be identical for a dimensionally homogeneous equation. Such equations are independent of the system of units.

Let us consider the equation, $V = \sqrt{2gH}$

Dimension of L.H.S. $= V = \frac{L}{T} = LT^{-1}$

Dimension of R.H.S. $= \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} = LT^{-1}$

Dimension of L.H.S. = Dimension of R.H.S. $= LT^{-1}$

$\therefore$ Equation $V = \sqrt{2gH}$ is dimensionally homogeneous. So it can be used in any system of units.

DIMENSIONAL AND MODEL ANALYSIS

▶ 12.4 METHODS OF DIMENSIONAL ANALYSIS

If the number of variable involved in a physical phenomenon are known, then the relation among the variables can be determined by the following two methods :

1. Rayleigh's method, and
2. Buckingham's π-theorem.

DIMENSIONAL AND MODEL ANALYSIS

12.4.2 Buckingham's π-Theorem.

Buckingham's π-theorem states as follows:

"If there are n variables (dependent and independent variables) in a dimensionally homogeneous equation and if these variables contain m fundamental dimensions (such as M, L, T, etc.) then the variables are arranged into (n-m) dimensionless terms. These dimensionless terms are called π-terms."

Mathematically, if any variable X_1, depends on independent variables, $X_2, X_3, X_4, \ldots X_n$; the functional equation may be written as:

$$X_1 = f(X_2, X_3, X_4, \ldots X_n) \qquad \ldots(7.3)$$

Eqn. (7.3) can also be written as:

$$f_1(X_1, X_2, X_3, \ldots X_n) = 0 \qquad \ldots(7.4)$$

It is a dimensionally homogeneous equation and contains n variables. If there are m fundamental dimensions, then according to Buckingham's π-theorem, it [eqn. (7.4)] can be written in terms of number of π-terms (dimensionless groups) in which number of π-terms is equal to $(n-m)$. Hence, eqn. (7.4) becomes as:

$$f_1(\pi_1, \pi_2, \pi_3 \ldots \pi_{n-m}) = 0 \qquad \ldots(7.5)$$

Each dimensionless π-term is formed by combining m variables out of the total n variables with *one of the remaining (n-m) variables* i.e. each π-term contains $(m + 1)$ variables. These m variables which appear repeatedly in each of π-terms are consequently called ***repeating variables*** and are chosen from among the variables such that they together *involve all the fundamental dimensions* and *they themselves do not form a dimensionless parameter.* Let in the above case X_2, X_3 and X_4 are the repeating variables if the fundamental dimensions m $(M, L, T) = 3$, then each term is written as:

$$\left.\begin{aligned} \pi_1 &= X_2^{a_1}.X_3^{b_1}.X_4^{c_1}.X_1 \\ \pi_2 &= \left(X_2^{a_2}.X_3^{b_2}.X_4^{c_2}.X_5\right) \\ &\vdots \\ &\vdots \\ \pi_{n-m} &= \left(X_2^{a_{n-m}}.X_3^{b_{n-m}}.X_4^{c_{n-m}}.X_n\right) \end{aligned}\right\} \qquad \ldots(7.6)$$

DIMENSIONAL AND MODEL ANALYSIS

where $a_1, b_1, c_1; a_2, b_2, c_2$ etc. are the constants, which are determined by considering dimensional homogeneity. These values are substituted in eqn. (7.6) and values of $\pi_1, \pi_2, \pi_3 \ldots \pi_{n-m}$ are obtained. These values of π's are substituted in eqn. (7.5). The final general equation for the phenomenon may then be obtained by expressing anyone of the π-terms as a function of the other as

$$\left.\begin{aligned} \pi_1 &= \phi(\pi_2, \pi_3, \pi_4, \ldots \pi_{n-m}) \\ \pi_2 &= \phi(\pi_1, \pi_3, \pi_4, \ldots \pi_{n-m}) \end{aligned}\right\} \quad \ldots(7.7)$$

12.4.3 Method of Selecting Repeating Variables. The number of repeating variables are equal to the number of fundamental dimensions of the problem. The choice of repeating variables is governed by the following considerations :

1. As far as possible, the dependent variable should not be selected as repeating variable.
2. The repeating variables should be choosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Variables with Geometric Property are

(*i*) Length, l (*ii*) d (*iii*) Height, H etc.

Variables with flow property are

(*i*) Velocity, V (*ii*) Acceleration etc.

Variables with fluid property : (*i*) μ, (*ii*) ρ, (*iii*) ω etc.

3. The repeating variables selected should not form a dimensionless group.
4. The repeating variables together must have the same number of fundamental dimensions.
5. No two repeating variables should have the same dimensions.

In most of fluid mechanics problems, the choice of repeating variables may be (*i*) d, v, ρ (*ii*) l, v, ρ or (*iii*) l, v, μ or (*iv*) d, v, μ.

DIMENSIONAL AND MODEL ANALYSIS

Example 7.9. *Using Buckingham's π-theorem, show that the velocity through a circular orifice is given by*

$$V = \sqrt{2gH}\,\phi\left[\frac{D}{H}, \frac{\mu}{\rho V H}\right]$$

where,

H = *Head causing flow,*

D = *Diameter of the orifice,*

μ = *Co-efficient of viscosity,*

ρ = *Mass density, and*

g = *Acceleration due to gravity.* **[GATE]**

Solution. V is a function of: H, D, μ, ρ and g

Mathematically, $V = f(H, D, \mu, \rho, g)$ ...(*i*)

or, $f_1(V, H, D, \mu, \rho, g) = 0$ ...(*ii*)

∴ Total number of variables, $n = 6$

Writing dimensions of each variable, we have:

$$V = LT^{-1},\ H = L,\ D = L,\ \mu = ML^{-1}T^{-1},\ \rho = ML^{-3},\ g = LT^{-2}$$

Thus, number of fundamental dimensions, $m = 3$

∴ Number of π-terms $= n - m = 6 - 3 = 3$

Eqn. (*ii*) can be written as:

$$f_1(\pi_1, \pi_2, \pi_3) = 0 \qquad ...(iii)$$

DIMENSIONAL AND MODEL ANALYSIS

Each π-term contains $(m + 1)$ variables, where $m = 3$ and is also equal to repeating variables. Choosing H, g, ρ as *repeating variables* (V being a dependent variable should not be chosen as repeating variable), we get three π-terms as:

$$\pi_1 = H^{a_1} . g^{b_1} \rho^{c_3} . V$$

$$\pi_2 = H^{a_2} . g^{b_2} \rho^{c_2} . D$$

$$\pi_3 = H^{a_3} . g^{b_3} \rho^{c_3} . \mu$$

π_1-term:

$$\pi_1 = H^{a_1} . g^{b_1} \rho^{c_1} . V$$

$$M^0L^0T^0 = L^{a_1} . (LT^{-2})^{b_1} . (ML^{-3})^{c_1} . (LT^{-1})$$

Equating the exponents of M, L and T respectively, we get:

For M: $\quad 0 = c_1$

For L: $\quad 0 = a_1 + b_1 - 3c_1 + 1$

For T: $\quad 0 = -2b_1 - 1$

$$\therefore \quad c_1 = 0; \; b_1 = -\frac{1}{2}$$

$$\text{and,} \quad a_1 = -b_1 + 3c_1 - 1 = \frac{1}{2} + 0 - 1 = -\frac{1}{2}$$

Substituting the values of a_1, b_1 and c_1 in π_1, we get:

$$\therefore \quad \pi_1 = H^{-1/2} . g^{-\frac{1}{2}} . \rho^0 . V = \frac{V}{\sqrt{gh}}$$

DIMENSIONAL AND MODEL ANALYSIS

π_2-term:

$$\pi_2 = H^{a_2} . g^{b_2} . \rho^{c_2} . D$$

$$M^0L^0T^0 = L^{a_2} . (LT^{-2})^{b_2} . (ML^{-3})^{c_2} . L$$

Equating the exponents of *M, L* and *T* respectively, we get:

For M: $0 = c_2$

For L: $0 = a_2 + b_2 - 3c_2 + 1$

For T: $0 = -2b_2$

$\therefore$ $c_2 = 0; b_2 = 0$

and, $a_2 = -b_2 + 3c_2 - 1 = -1$

Substituting the values of a_2, b_2, and c_2 in π_2, we get:

$$\pi_2 = H^{-1} . g^0 . \rho^0 . D = \frac{D}{H}$$

DIMENSIONAL AND MODEL ANALYSIS

π_3-term:

$$\pi_3 = H^{a_3} . g^{b_3} . \rho^{c_3} . \mu$$

$$M^0L^0T^0 = L^{a_3} . (LT^{-2})^{b_3} . (ML^{-3})^{c_3} . ML^{-1}T^{-1}$$

Equating the exponents of M, L and T respectively, we get:

For M: $0 = c_3 + 1$

For L: $0 = a_3 + b_3 - 3c_3 - 1$

For T: $0 = -2b_3 - 1$

$\therefore$ $c_3 = -1; b_3 = -\frac{1}{2}$

and, $a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1 = -\frac{3}{2}$

Substituting the values of a_3, b_3, and c_3 in π_3, we get:

$$\pi_3 = H^{-3/2} \cdot g^{-\frac{1}{2}} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{H^{3/2}\rho\sqrt{g}}$$

$$= \frac{\mu}{H\rho\sqrt{gH}} = \frac{\mu V}{H\rho V\sqrt{gH}} \quad \text{(Multiply and divide by } V\text{)}$$

$$= \frac{\mu}{H\rho V} \cdot \pi_1 \quad \left(\because \frac{V}{\sqrt{gH}} = \pi_1\right)$$

DIMENSIONAL AND MODEL ANALYSIS

Substituting the values of π_1, π_2 and π_3 in eqn. (*iii*), we get:

$$f_1\left(\frac{V}{\sqrt{gH}}, \frac{D}{H}, \frac{\mu}{HlV} . \pi_1\right) = 0$$

or,
$$\frac{V}{\sqrt{gH}} = \phi\left[\frac{D}{H}, \frac{\mu}{H\rho V}.\pi_1\right]$$

or,
$$V = \sqrt{2gH}\,\phi\left[\frac{D}{H}, \frac{\mu}{\rho VH}\right]$$ ...**Proved.**

(*Multiplying or dividing by any constant does not change the character of* π-*terms*).

DIMENSIONAL AND MODEL ANALYSIS

▶ 12.5 MODEL ANALYSIS

The **model** is the small scale replica of the actual structure or machine. The actual structure or machine is called **Prototype**.

▶ 12.6 SIMILITUDE-TYPES OF SIMILARITIES

1. Geometric Similarity, 2. Kinematic Similarity, and 3. Dynamic Similarity.

DIMENSIONAL AND MODEL ANALYSIS

1. Geometric Similarity. The geometric similarity is said to exist between the model and the prototype. The ratio of all corresponding linear dimension in the model and prototype are equal.

Let L_m = Length of model, b_m = Breadth of model,

D_m = Diameter of model, A_m = Area of model,

$\forall_m$ = Volume of model,

and $L_P, b_P, D_P, A_P, \forall_P$ = Corresponding values of the prototype.

For geometric similarity between model and prototype, we must have the relation,

$$\frac{L_P}{L_m} = \frac{b_P}{b_m} = \frac{D_P}{D_m} = L_r \quad \text{...(12.6)}$$

where L_r is called the scale ratio.

For area's ratio and volume's ratio the relation should be as given below :

$$\frac{A_P}{A_m} = \frac{L_P \times b_P}{L_m \times b_m} = L_r \times L_r = L_r^2 \quad \text{...(12.7)}$$

and

$$\frac{\forall_P}{\forall_m} = \left(\frac{L_P}{L_m}\right)^3 = \left(\frac{b_P}{b_m}\right)^3 = \left(\frac{D_P}{D_m}\right)^3 \quad \text{...(12.8)}$$

DIMENSIONAL AND MODEL ANALYSIS

2. Kinematic Similarity. Kinematic similarity means the similarity of motion between model and prototype. Thus kinematic similarity is said to exist between the model and the prototype if the ratios of the velocity and acceleration at the corresponding points in the model and at the corresponding points in the prototype are the same. Since velocity and acceleration are vector quantities, hence not only the ratio of magnitude of velocity and acceleration at the corresponding points in model and prototype should be same ; but the directions of velocity and accelerations at the corresponding points in the model and prototype also should be parallel.

Let V_{P_1} = Velocity of fluid at point 1 in prototype,

V_{P_2} = Velocity of fluid at point 2 in prototype,

a_{P_1} = Acceleration of fluid at point 1 in prototype,

a_{P_2} = Acceleration of fluid at point 2 in prototype, and

$V_{m_1}, V_{m_2}, a_{m_1}, a_{m_2}$ = Corresponding values at the corresponding points of fluid velocity and acceleration in the model.

For kinematic similarity, we must have

$$\frac{V_{P_1}}{V_{m_1}} = \frac{V_{P_2}}{V_{m_2}} = V_r \quad \text{...(12.9)}$$

where V_r is the velocity ratio.

For acceleration, we must have $\frac{a_{P_1}}{a_{m_1}} = \frac{a_{P_2}}{a_{m_2}} = a_r$ ...(12.10)

where a_r is the acceleration ratio.

Also the directions of the velocities in the model and prototype should be same.

DIMENSIONAL AND MODEL ANALYSIS

3. Dynamic Similarity. Dynamic similarity means the similarity of forces between the model and prototype. Thus dynamic similarity is said to exist between the model and the prototype if the ratios of the corresponding forces acting at the corresponding points are equal. Also the directions of the corresponding forces at the corresponding points should be same.

Let $(F_i)_P$ = Inertia force at a point in prototype,

$(F_v)_P$ = Viscous force at the point in prototype,

$(F_g)_P$ = Gravity force at the point in prototype,

and $(F_i)_m$ $(F_v)_m$, $(F_g)_m$ = Corresponding values of forces at the corresponding point in model.

Then for dynamic similarity, we have

$$\frac{(F_i)_P}{(F_i)_m} = \frac{(F_v)_P}{(F_v)_m} = \frac{(F_g)_P}{(F_g)_m} \dots = F_r, \text{ where } F_r \text{ is the force ratio.}$$

Also the directions of the corresponding forces at the corresponding points in the model and prototype should be same.

DIMENSIONAL AND MODEL ANALYSIS

▶ 12.8 DIMENSIONLESS NUMBERS

Dimensionless numbers are those numbers which are obtained by dividing the inertia force by viscous force or gravity force or pressure force or surface tension force or elastic force. As this is a ratio of one force to the other force, it will be a dimensionless number. These dimensionless numbers are also called non-dimensional parameters. The followings are the important dimensionless numbers :

1. Reynold's number,
2. Froude's number,
3. Euler's number,
4. Weber's number,
5. Mach's number.

12.8.1 Reynold's Number (R_e). It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. The expression for Reynold's number is obtained as

Inertia force (F_i) = Mass × Acceleration of flowing fluid

$$= \rho \times \text{Volume} \times \frac{\text{Velocity}}{\text{Time}} = \rho \times \frac{\text{Volume}}{\text{Time}} \times \text{Velocity}$$

$$= \rho \times AV \times V \quad \{\because \text{ Volume per sec} = \text{Area} \times \text{Velocity} = A \times V\}$$

$$= \rho A V^2 \qquad ...(12.11)$$

Viscous force (F_v)

$$= \text{Shear stress} \times \text{Area} \qquad \left\{\because \tau = \mu \frac{du}{dy} \quad \therefore \text{ Force} = \tau \times \text{Area}\right\}$$

$$= \tau \times A$$

$$= \left(\mu \frac{du}{dy}\right) \times A = \mu \, . \, \frac{V}{L} \times A \qquad \left\{\because \frac{du}{dy} = \frac{V}{L}\right\}$$

By definition, Reynold's number,

$$R_e = \frac{F_i}{F_v} = \frac{\rho A V^2}{\mu \, . \, \frac{V}{L} \times A} = \frac{\rho V L}{\mu}$$

$$= \frac{V \times L}{(\mu / \rho)} = \frac{V \times L}{\nu} \qquad \left\{\because \frac{\mu}{\rho} = \nu = \text{Kinematic viscosity}\right\}$$

In case of pipe flow, the linear dimension L is taken as diameter, d. Hence Reynold's number for pipe flow,

$$R_e = \frac{V \times d}{\nu} \quad \text{or} \quad \frac{\rho V d}{\mu}. \qquad ...(12.12)$$

DIMENSIONAL AND MODEL ANALYSIS

12.8.2 Froude's Number (F_e). The Froude's number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravity force. Mathematically, it is expressed as

$$F_e = \sqrt{\frac{F_i}{F_g}}$$

where F_i from equation (12.11) $= \rho A V^2$

and F_g = Force due to gravity

= Mass × Acceleration due to gravity

$= \rho \times \text{Volume} \times g = \rho \times L^3 \times g$ $\{\because \text{ Volume} = L^3\}$

$= \rho \times L^2 \times L \times g = \rho \times A \times L \times g$ $\{\because L^2 = A = \text{Area}\}$

$$\therefore \quad F_e = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho A V^2}{\rho A L g}} = \sqrt{\frac{V^2}{Lg}} = \frac{V}{\sqrt{Lg}} \quad \text{...(12.13)}$$

DIMENSIONAL AND MODEL ANALYSIS

12.8.3 Euler's Number (E_u). It is defined as the square root of the ratio of the inertia force of a flowing fluid to the pressure force. Mathematically, it is expressed as

$$E_u = \sqrt{\frac{F_i}{F_P}}$$

where F_P = Intensity of pressure × Area = $p \times A$

and $F_i = \rho A V^2$

$$\therefore \qquad E_u = \sqrt{\frac{\rho A V^2}{p \times A}} = \sqrt{\frac{V^2}{p / \rho}} = \frac{V}{\sqrt{p / \rho}} \qquad \text{...(12.14)}$$

DIMENSIONAL AND MODEL ANALYSIS

12.8.4 Weber's Number (W_e). It is defined as the square root of the ratio of the inertia force of a flowing fluid to the surface tension force. Mathematically, it is expressed as

Weber's Number, $$W_e = \sqrt{\frac{F_i}{F_s}}$$

where F_i = Inertia force = $\rho A V^2$

and F_s = Surface tension force

= Surface tension per unit length × Length = $\sigma \times L$

$$\therefore \quad W_e = \sqrt{\frac{\rho A V^2}{\sigma \times L}} = \sqrt{\frac{\rho \times L^2 \times V^2}{\sigma \times L}} \qquad \{\because \quad A = L^2\}$$

$$= \sqrt{\frac{\rho L \times V^2}{\sigma}} = \sqrt{\frac{V^2}{\sigma / \rho L}} = \frac{V}{\sqrt{\sigma / \rho L}}. \qquad ...(12.15)$$

DIMENSIONAL AND MODEL ANALYSIS

12.8.5 Mach's Number (M). Mach's number is defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force. Mathematically, it is defined as

$$M = \sqrt{\frac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\frac{F_i}{F_e}}$$

where $F_i = \rho A V^2$

and $F_e = \text{Elastic force} = \text{Elastic stress} \times \text{Area}$

$= K \times A = K \times L^2$ $\{\because \; K = \text{Elastic stress}\}$

$$\therefore \quad M = \sqrt{\frac{\rho A V^2}{K \times L^2}} = \sqrt{\frac{\rho \times L^2 \times V^2}{K \times L^2}} = \sqrt{\frac{V^2}{K/\rho}} = \frac{V}{\sqrt{K/\rho}}$$

But $\sqrt{\frac{K}{\rho}} = C = \text{Velocity of sound in the fluid}$

$$\therefore \quad M = \frac{V}{C}. \qquad \text{...(12.16)}$$

DIMENSIONAL AND MODEL ANALYSIS

12.9.1 Reynold's Model Law. Reynold's model law is the law in which models are based on Reynold's number. Models based on Reynold's number includes :

(*i*) Pipe flow

(*ii*) Resistance experienced by sub-marines, airplanes, fully immersed bodies etc.

$$[R_e]_m = [R_e]_P \text{ or } \frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_P V_P L_P}{\mu_P}$$

or $$\frac{\rho_P \,.\, V_P \,.\, L_P}{\rho_m \,.\, V_m \,.\, L_m} \times \frac{1}{\dfrac{\mu_P}{\mu_m}} = 1 \quad \text{or} \quad \frac{\rho_r \,.\, V_r \,.\, L_r}{\mu_r} = 1$$

DIMENSIONAL AND MODEL ANALYSIS

12.9.2 Froude Model Law. Froude model law is the law in which the models are based on Froude number which means for dynamic similarity between the model and prototype, the Froude number for both of them should be equal. Froude model law is applicable when the gravity force is only predominant force which controls the flow in addition to the force of inertia. Froude model law is applied in the following fluid flow problems :

1. Free surface flows such as flow over spillways, weirs, sluices, channels etc.,
2. Flow of jet from an orifice or nozzle,
3. Where waves are likely to be formed on surface,
4. Where fluids of different densities flow over one another.

DIMENSIONAL AND MODEL ANALYSIS

$$(F_e)_{model} = (F_e)_{prototype} \text{ or } \frac{V_m}{\sqrt{g_m L_m}} = \frac{V_P}{\sqrt{g_P L_P}} \qquad \text{...(12.18)}$$

If the tests on the model are performed on the same place where prototype is to operate, then $g_m = g_P$ and equation (12.18) becomes as

$$\frac{V_m}{\sqrt{L_m}} = \frac{V_P}{\sqrt{L_P}} \qquad \text{...(12.19)}$$

or

$$\frac{V_m}{V_P} \times \frac{1}{\sqrt{\frac{L_m}{L_P}}} = 1$$

$$\frac{V_P}{V_m} = \sqrt{\frac{L_P}{L_m}} = \sqrt{L_r} \qquad \left\{\because \frac{L_P}{L_m} = L_r\right\}$$

where L_r = Scale ratio for length

$\frac{V_P}{V_m} = V_r$ = Scale ratio for velocity.

$$\therefore \qquad \frac{V_P}{V_m} = V_r = \sqrt{L_r}\,. \qquad \text{...(12.20)}$$

Scale ratios for various physical quantities based on Froude model law are :

DIMENSIONAL AND MODEL ANALYSIS

(*a*) **Scale ratio for time**

As $$\text{time} = \frac{\text{Length}}{\text{Velocity}},$$

then ratio of time for prototype and model is

$$T_r = \frac{T_P}{T_m} = \frac{\left(\dfrac{L}{V}\right)_P}{\left(\dfrac{L}{V}\right)_m} = \frac{\dfrac{L_P}{V_P}}{\dfrac{L_m}{V_m}} = \frac{L_P}{L_m} \times \frac{V_m}{V_P} = L_r \times \frac{1}{\sqrt{L_r}} \qquad \left\{\because \frac{V_P}{V_m} = \sqrt{L_r}\right\}$$

$$= \sqrt{L_r}. \qquad \text{...(12.21)}$$

(*b*) **Scale ratio for acceleration**

$$\text{Acceleration} = \frac{V}{T}$$

$$\therefore \qquad a_r = \frac{a_P}{a_m} = \frac{\left(\dfrac{V}{T}\right)_P}{\left(\dfrac{V}{T}\right)_m} = \frac{V_P}{T_P} \times \frac{T_m}{V_m} = \frac{V_P}{V_m} \times \frac{T_m}{T_P}$$

DIMENSIONAL AND MODEL ANALYSIS

$$= \sqrt{L_r} \times \frac{1}{\sqrt{L_r}} \qquad \left\{\because \frac{V_P}{V_m} = \sqrt{L_r}, \frac{T_P}{T_m} = \sqrt{L_r}\right\}$$

$$= 1. \qquad \text{...(12.22)}$$

(c) **Scale ratio for discharge**

$$Q = A \times V = L^2 \times \frac{L}{T} = \frac{L^3}{T}$$

$$\therefore \quad Q_r = \frac{Q_P}{Q_m} = \frac{\left(\frac{L^3}{T}\right)_P}{\left(\frac{L^3}{T}\right)_m} = \left(\frac{L_P}{L_m}\right)^3 \times \left(\frac{T_m}{T_P}\right) = L_r^{\,3} \times \frac{1}{\sqrt{L_r}} = L_r^{\,2.5} \qquad \text{...(12.23)}$$

DIMENSIONAL AND MODEL ANALYSIS

(*d*) **Scale ratio for force**

As $$\text{Force} = \text{Mass} \times \text{Acceleration} = \rho L^3 \times \frac{V}{T} = \rho L^2 \,.\, \frac{L}{T} \,.\, V = \rho L^2 V^2$$

$\therefore$ Ratio for force, $$F_r = \frac{F_P}{F_m} = \frac{\rho_P L_P^2 V_P^2}{\rho_m L_m^2 V_m^2} = \frac{\rho_P}{\rho_m} \times \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2.$$

If the fluid used in model and prototype is same, then

$$\frac{\rho_P}{\rho_m} = 1 \quad \text{or} \quad \rho_P = \rho_m$$

and hence $$F_r = \left(\frac{L_P}{L_m}\right)^2 \times \left(\frac{V_P}{V_m}\right)^2 = L_r^2 \times \left(\sqrt{L_r}\right)^2 = L_r^2 \,.\, L_r = L_r^3. \qquad \text{...(12.24)}$$

DIMENSIONAL AND MODEL ANALYSIS

(*e*) Scale ratio for pressure intensity

As $$p = \frac{\text{Force}}{\text{Area}} = \frac{\rho L^2 V^2}{L^2} = \rho V^2$$

∴ Pressure ratio, $$p_r = \frac{p_P}{p_m} = \frac{\rho_P V_P^2}{\rho_m V_m^2}$$

If fluid is same, then $\rho_P = \rho_m$

∴ $$p_r = \frac{V_P^2}{V_m^2} = \left(\frac{V_P}{V_m}\right)^2 = L_r.$$

(*f*) Scale ratio for work, energy, torque, moment etc.

$$\text{Torque} = \text{Force} \times \text{Distance} = F \times L$$

∴ Torque ratio, $$T_r^* = \frac{T_P^*}{T_m^*} = \frac{(F \times L)_P}{(F \times L)_m} = F_r \times L_r = L_r^3 \times L_r = L_r^4.$$

(*g*) Scale ratio for power

As Power = Work per unit time

DIMENSIONAL AND MODEL ANALYSIS

$$= \frac{F \times L}{T}$$

∴ Power ratio,

$$P_r = \frac{P_P}{P_m} = \frac{\dfrac{F_P \times L_P}{T_P}}{\dfrac{F_m \times L_m}{T_m}} = \frac{F_P}{F_m} \times \frac{L_P}{L_m} \times \frac{1}{\dfrac{T_P}{T_m}}$$

$$= F_r \,.\, L_r \,.\, \frac{1}{T_r} = L_r^{\,3} \,.\, L_r \,.\, \frac{1}{\sqrt{L_r}} = L^{3.5}. \qquad \text{...(12.27)}$$

DIMENSIONAL AND MODEL ANALYSIS

Problem 12.21 *In the model test of a spillway the discharge and velocity of flow over the model were 2 m³/s and 1.5 m/s respectively. Calculate the velocity and discharge over the prototype which is 36 times the model size.*

Solution. Given :

Discharge over model, $Q_m = 2 \text{ m}^3/\text{s}$

Velocity over model, $V_m = 1.5 \text{ m/s}$

Linear scale ratio, $L_r = 36.$

For dynamic similarity, Froude model law is used. Using equation (12.20), we have

$$\frac{V_P}{V_m} = \sqrt{L_r} = \sqrt{36} = 6.0$$

$\therefore$ V_P = Velocity over prototype $= V_m \times 6.0 = 1.5 \times 6.0 =$ **9 m/s. Ans.**

For discharge, using equation (12.23), we get

$$\frac{Q_P}{Q_m} = L_r^{2.5} = (36)^{2.5}.$$

$\therefore$ $Q_P = Q_m \times (36)^{2.5} = 2 \times 36^{2.5} =$ **15552 m³/s. Ans.**

DIMENSIONAL AND MODEL ANALYSIS

$= \rho_r A_r V_r = \rho_r \cdot L_r^2 \cdot V_r.$

Problem 12.15 *A pipe of diameter 1.5 m is required to transport an oil of sp. gr. 0.90 and viscosity 3×10^{-2} poise at the rate of 3000 litre/s. Tests were conducted on a 15 cm diameter pipe using water at 20°C. Find the velocity and rate of flow in the model. Viscosity of water at 20°C = 0.01 poise.*

Solution. Given :

Dia. of prototype, $D_P = 1.5$ m

Viscosity of fluid, $\mu_P = 3 \times 10^{-2}$ poise

Q for prototype, $Q_P = 3000$ lit/s $= 3.0\ m^3/s$

Sp. gr. of oil, $S_P = 0.9$

∴ Density of oil, $\rho_P = S_P \times 1000 = 0.9 \times 1000 = 900\ kg/m^3$

Dia. of the model, $D_m = 15$ cm $= 0.15$ m

Viscosity of water at 20°C $= .01$ poise $= 1 \times 10^{-2}$ poise or $\mu_m = 1 \times 10^{-2}$ poise

Density of water or $\rho_m = 1000\ kg/m^3$.

DIMENSIONAL AND MODEL ANALYSIS

For pipe flow, the dynamic similarity will be obtained if the Reynold's number in the model and prototype are equal

Hence using equation (12.17), $\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_P V_P D_P}{\mu_P}$ {For pipe, linear dimension is D}

$$\therefore \quad \frac{V_m}{V_P} = \frac{\rho_P}{\rho_m} \cdot \frac{D_P}{D_m} \cdot \frac{\mu_m}{\mu_P}$$

$$= \frac{900}{1000} \times \frac{1.5}{0.15} \times \frac{1 \times 10^{-2}}{3 \times 10^{-2}} = \frac{900}{1000} \times 10 \times \frac{1}{3} = 3.0$$

But $$V_P = \frac{\text{Rate of flow in prototype}}{\text{Area of prototype}} = \frac{3.0}{\frac{\pi}{4}(D_P)^2} = \frac{3.0}{\frac{\pi}{4}(1.5)^2}$$

$$= \frac{3.0 \times 4}{\pi \times 2.25} = 1.697 \text{ m/s}$$

$$\therefore \quad V_m = 3.0 \times V_P = 3.0 \times 1.697 = \mathbf{5.091 \text{ m/s. Ans.}}$$

Rate of flow through model, $Q_m = A_m \times V_m = \frac{\pi}{4}(D_m)^2 \times V_m = \frac{\pi}{4}(0.15)^2 \times 5.091 \text{ m}^3\text{/s}$

$$= 0.0899 \text{ m}^3\text{/s} = 0.0899 \times 1000 \text{ lit/s} = \mathbf{89.9 \text{ lit/s. Ans.}}$$

LAMINAR AND TURBULANT FLOW

10.2. REYNOLDS EXPERIMENT

Osborne Reynolds in 1883, with the help of a simple experiment (see Fig. 10.1), demonstrated the existence of the following two types of flows:

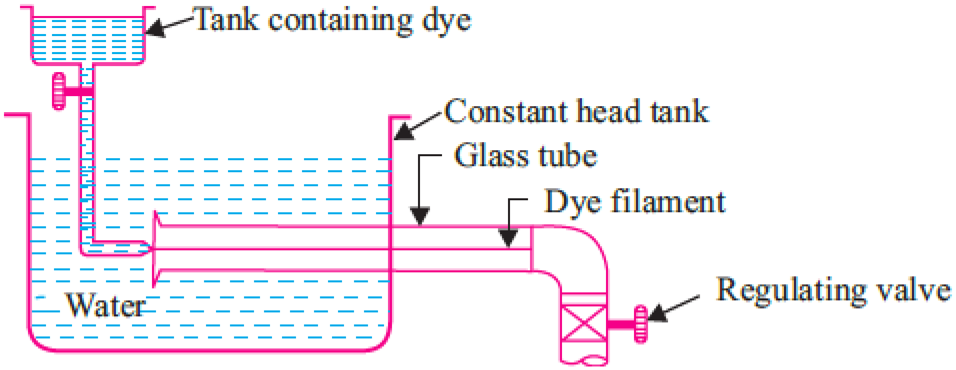


Fig.10.1. Reynolds apparatus.

1. Laminar flow (Reynolds number, $Re < 2000$)
2. Turbulent flow (Reynolds number, $Re > 4000$)

(*Re* between 2000 and 4000 indicates *transition from laminar to turbulent flow*)

LAMINAR AND TURBULANT FLOW

Observations made:

1. When the *velocity* of flow was *low*, the dye remained in the form of a *straight* and *stable filament* passing through the glass tube so steadily that it scarecely seemed to be in motion. This was a case of **laminar flow** as shown in Fig. 10.2 (*a*).
2. With the increase of velocity a critical state was reached at which the dye filament showed irregularities and began to waver (see Fig. 10.2 *b*). This shows that the flow is no longer a laminar one. This was a **transitional state.**
3. With further increase in velocity of flow the fluctuations in the filament of dye became more intense and ultimately the dye diffused over the entire cross-section of the tube, due to the intermingling of the particles of the flowing fluid. This was the case of a **turbulent flow** as shown in Fig. 10.2 (*c*).

On the basis of his experiment Reynolds discovered that:

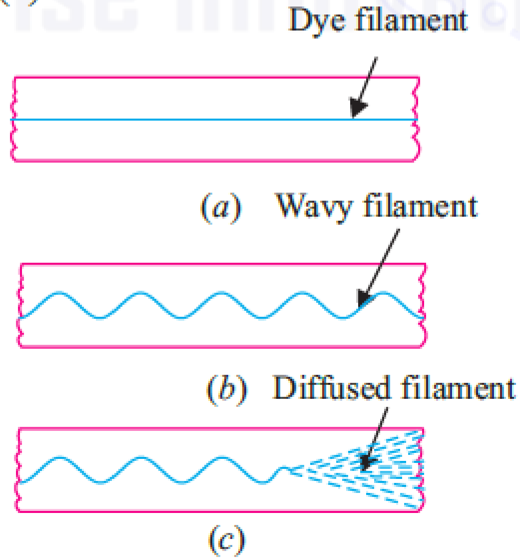


Fig. 10.2. Appearance of dye filament in (*a*) laminar flow, (*b*) transition, and (*c*) turbulent flow.

LAMINAR AND TURBULANT FLOW

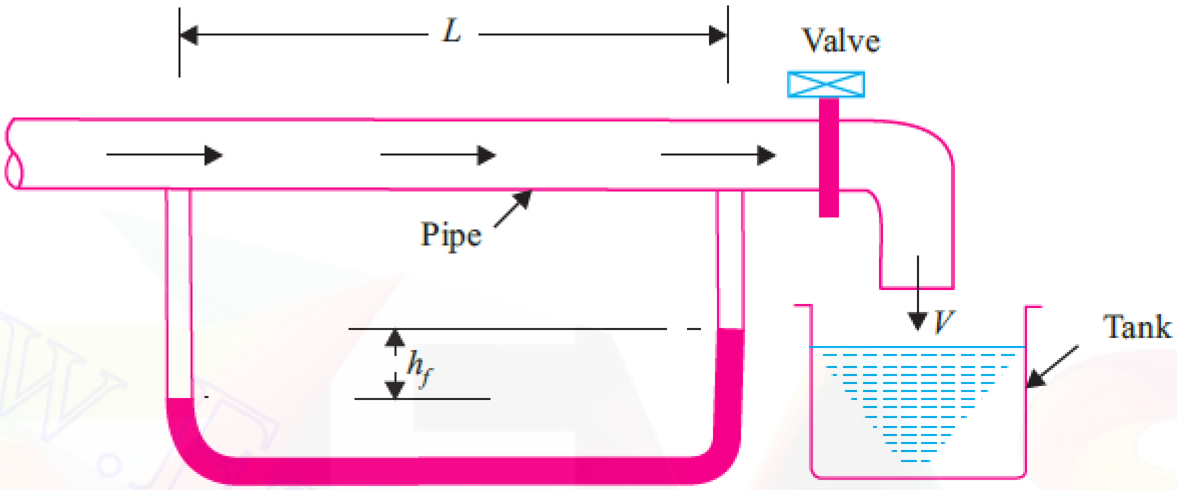
(*i*) In case of **laminar flow:** The loss of pressure head ∝ velocity.

(*ii*) In case of **turbulent flow:** The loss of head is approximately ∝ V^2

[More exactly the loss of head ∝ V^n where n varies from 1.75 to 2.0]

Fig. 10.3 shows the apparatus used by Reynolds for estimating the loss of head in a pipe by neasuring the pressure difference over a known length of the pipe.

(*i*) The velocity of water in the pipe was determined by measuring the volume of water (Q) collected in the tank over a known period of time $\left(V = \frac{Q}{A}\right.$, where A is the area of cross-section of the pipe.)



LAMINAR AND TURBULANT FLOW

(*ii*) The velocity of flow (V) was changed and corresponding values of h_f (loss of head) were obtained.

(*iii*) A graph was plotted between V (velocity of flow) and h_f (loss of head). Such a graph is shown in Fig. 10.4. It may be seen from the graph that:

(*a*) At *low velocities* the curve is a straight line, indicating that the h_f (loss of head) is *directly proportional to velocity*—the flow is **laminar** (or viscous).

(*b*) At *higher velocities* the curve is parabolic; in this range $h_f \alpha V^n$, where the value of n lies between 1.75 to 2.0 — the flow is **turbulent.**

(*c*) In the intermediate region, there is a transition zone. This is shown by *dotted line*.

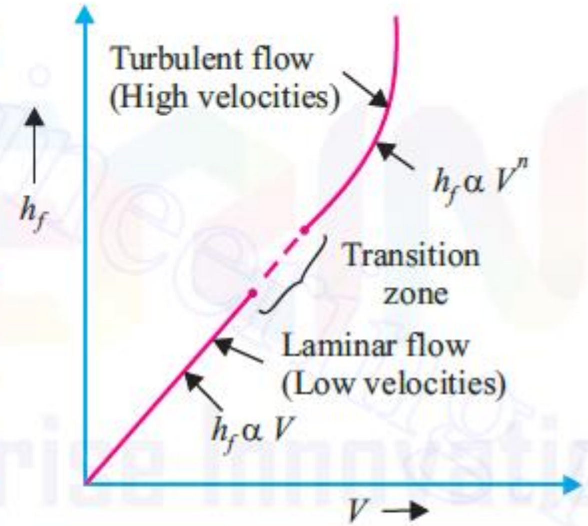


Fig. 10.4

LAMINAR AND TURBULANT FLOW

i.e. Reynolds number $Re = \frac{\rho V D}{\mu}$

It may also be expressed as:

$$Re = \frac{VD}{\nu}$$

where, ν = Kinematic viscosity $\left(= \frac{\mu}{\rho} \right)$

when, $Re < 2000$ ... the flow is *laminar* (or viscous)

$Re > 4000$ ... the flow is *turbulent*.

Re between 2000 and 4000 ... the flow is *unpredictable*.